MIXED-INTEGER DAE OPTIMAL CONTROL PROBLEMS: NECESSARY CONDITIONS AND BOUNDS

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Abstract. We are interested in the optimal control of dynamic processes that can be described by Differential Algebraic Equations (DAEs) and that include integer restrictions on some or all of the control functions. We assume the DAE system to be of index 1. In our study we consider necessary conditions of optimality for this specific case of a hybrid system and results on lower bounds that are important in an algorithmic setting. Both results generalize previous work for the case of Ordinary Differential Equations (ODE). Interestingly, the proofs for both analytical results are based on constructive elements to obtain integer controls from reformulations or relaxations to purely continuous control functions. These constructive elements can also be used for an efficient numerical calculation of optimal solutions. We illustrate the theoretical results by means of a mixed–integer nonlinear optimal control benchmark problem with algebraic variables.

Key words. mixed integer programming, nonlinear programming, DAE, switched systems

1. Introduction. Technical or economical processes often involve *discrete* control variables, which are used to model finitely many decisions, discrete resources, or switching structures like gear shifts in a car or operating modes of a device. This leads to optimal control problems with non-convex and partly discrete control set $\mathcal{U}$. More specifically, some of the control variables may still assume any real value within a given convex set with non-empty interior, those are called *continuous-valued control variables* in the sequel, while other control variables are restricted to a finite set of values, those are called *discrete control variables* in the sequel.

An optimal control problem involving continuous-valued and discrete control variables is called *mixed-integer optimal control problem* (MIOCP). Mixed-integer optimal control is a field of increasing importance and practical applications can be found in [15, 11, 13, 34, 19]. For a web-site of further benchmark problems please refer to [27] and the corresponding paper [30].

An approach to solve mixed-integer optimal control problems is by exploiting necessary optimality conditions. A proof for index-one DAEs will be provided in Section 3. The proof exploits an idea of Dubovitskii and Milyutin, see [8, 7], [14, p. 95], [16, p. 148], who used a time transformation to transform the mixed-integer optimal control problem into an equivalent optimal control problem without discrete control variables. Necessary conditions are then obtained by applying suitable local minimum principles to the transformed problem. The result are necessary conditions in terms of *global minimum principles.*

A global minimum principle for disjoint control sets and (noncontinuous) ordinary differential equations (ODEs) has been formulated and solved numerically via the newly developed method of *Competing Hamiltonians* in the work of Bock and Longman, [2, 3, 20]. To our knowledge this was the first time that a global minimum principle has been applied to solve a MIOCP.

Global minimum principles for DAE optimal control problems can be found in [26] for Hessenberg DAE optimal control problems, in [5] for semi-explicit index one DAEs, in [6] for implicit control systems, in [1] for quasilinear DAEs, in [21] for

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nonlinear DAEs of arbitrary index, and in [36] for switched ODEs.

The global minimum principle can be exploited numerically using an indirect *first optimize, then discretize* approach, but a very good initial guess of the problem's switching structure is needed. Such an initial guess is often not available for practical applications.

The time transformation of Dubovitskii and Milyutin will be used in Section 3 as a theoretical tool to prove the global minimum principle. Interestingly, the same variable time transformation can be used numerically to solve mixed-integer optimal control problems, see [23, 24, 37, 34, 13], time optimal control problems, see [23], and singular optimal control problems, see [35]. A method for solving nonlinear mixed-integer programming problems based on a suitable formulation of an equivalent optimal control problem was introduced in [22].

An alternative approach based on a partial outer convexification, relaxation, and control grid adaptivity has been proposed in [28, 33, 31]. Extensions include the explicit treatment of combinatorial [32] and vanishing constraints [18]. A crucial ingredient are tight lower bounds. An extension of an important result to the DAE case is deduced in Section 4. Again, parts of the proof are constructive in the sense that they provide discrete control values.

We illustrate the global minimum principle for MIOCPs in DAE in Section 5 and close with a summary.

Throughout, $L_\infty^n(\mathcal{I})$ denotes the Banach space of essentially bounded n-vector functions on the compact interval $\mathcal{I} \subset \mathbb{R}$ and $W_{1,\infty}^n(\mathcal{I})$ denotes the Banach space of absolutely continuous n-vector functions on the compact interval $\mathcal{I}$ with essentially bounded first derivative.

2. Time Transformation. In this section we distinguish between continuous-valued controls u with values in the closed convex set $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ with $\operatorname{int}(\mathcal{U}) \neq 0$ and discrete controls v with values in the discrete finite set

$$\mathcal{V} := \{v^1, \ldots, v^{n_\omega}\} \qquad v^i \in \mathbb{R}^{n_v},\ n_\omega \in \mathbb{N}. \tag{2.1}$$

We investigate

PROBLEM 2.1 (Mixed-Integer Optimal Control Problem (MIOCP)). *Let $\mathcal{I} := [t_0, t_f]$ be a non-empty compact time interval with $t_0 < t_f$ fixed. Let*

$$\begin{aligned}
\varphi &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R},\\
f &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \longrightarrow \mathbb{R}^{n_x},\\
g &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \longrightarrow \mathbb{R}^{n_y},\\
s &: \mathbb{R}^{n_x} \longrightarrow \mathbb{R}^{n_x},\\
\psi &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R}^{n_\psi}
\end{aligned}$$

be sufficiently smooth functions, $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ closed and convex with non-empty interior, and $\mathcal{V}$ as in (2.1).

Minimize the objective function

$$\varphi(x(t_0), x(t_\mathrm{f}))$$

with respect to $x \in W_{1,\infty}^{n_x}(\mathcal{I})$, $y \in L_\infty^{n_y}(\mathcal{I})$, $u \in L_\infty^{n_u}(\mathcal{I})$, $v \in L_\infty^{n_v}(\mathcal{I})$ subject to the semi-explicit DAE

$$\begin{aligned}
\dot{x}(t) &= f(x(t), y(t), u(t), v(t)) \quad \text{a.e. in } \mathcal{I},\\
0_{\mathbb{R}^{n_y}} &= g(x(t), y(t), u(t), v(t)) \quad \text{a.e. in } \mathcal{I},
\end{aligned}$$

the state constraint

$$s(x(t)) \leq 0_{\mathbb{R}^{n_s}},$$

the boundary condition

$$\psi(x(t_0), x(t_{\mathrm{f}})) = 0_{\mathbb{R}^{n_\psi}},$$

and the set constraints

$$\begin{aligned} u(t) &\in \mathcal{U} \quad \text{a.e. in } \mathcal{I}, \\ v(t) &\in \mathcal{V} \quad \text{a.e. in } \mathcal{I}. \end{aligned}$$

■

The variable time transformation method is based on a discretization. For simplicity only equally spaced grids are discussed. Let the *major grid*

$$\mathbb{G}_N := \{t_i = t_0 + ih \mid i = 0, \ldots, N\}, \quad h = \frac{t_{\mathrm{f}} - t_0}{N}$$

with $N \in \mathbb{N}$ intervals be given. Each major grid interval is subdivided into n_ω equally spaced subintervals, where n_ω denotes the number of values in the discrete control set $\mathcal{V}$ in (2.1). This leads to the *minor grid*

$$\mathbb{G}_{N,n_\omega} := \{\tau_{i,j} = t_i + j\frac{h}{n_\omega} \mid j = 0, \ldots, n_\omega,\ i = 0, \ldots, N-1\}.$$

On the minor grid define the fixed and piecewise constant function

$$v_{\mathbb{G}_{N,n_\omega}}(\tau) := v^j \quad \text{for } \tau \in [\tau_{i,j-1}, \tau_{i,j}),\ i = 0, \ldots, N-1,\ j = 1, \ldots, n_\omega. \tag{2.2}$$

Consider the time transformation

$$t(\tau) := t_0 + \int_{t_0}^{\tau} w(r)dr, \quad \tau \in \mathcal{I},$$

with

$$\int_{t_0}^{t_{\mathrm{f}}} w(r)dr = t_{\mathrm{f}} - t_0 \quad \text{and} \quad w(\tau) \geq 0 \quad \text{for almost every } \tau \in \mathcal{I}. \tag{2.3}$$

The inverse mapping is defined by

$$\tau(t) := \inf\{\tau \in \mathcal{I} \mid t(\tau) = t\}. \tag{2.4}$$

The time transformation controls the length of the intervals $[t(\tau_{i,j}), t(\tau_{i,j+1})]$ by proper choice of w according to

$$\int_{\tau_{i,j}}^{\tau_{i,j+1}} w(\tau)d\tau = t(\tau_{i,j+1}) - t(\tau_{i,j}).$$

Note that the time transformation maps $\mathcal{I}$ onto itself but changes the speed of running through this interval. In particular, it holds

$$\frac{dt}{d\tau}(\tau) = w(\tau) \quad \text{for } \tau \in \mathcal{I},$$

and the interval $[t(\tau_{i,j}), t(\tau_{i,j+1})]$ shrinks to the point $\{t(\tau_{i,j})\}$ if

$$w(\tau) = 0 \quad \text{in } [\tau_{i,j}, \tau_{i,j+1}).$$

State constraints $s(x(t)) \leq 0_{\mathbb{R}^{n_s}}$ will be evaluated at major grid points t_i, $i = 0, \ldots, N$, only, and hence we impose the additional constraints

$$\int_{t_i}^{t_{i+1}} w(\tau)d\tau = t_{i+1} - t_i = h, \quad i = 0, \ldots, N-1, \tag{2.5}$$

which ensure that the transformed time points $t_i = t(t_i)$, $i = 1, \ldots, N$, are fixed points. Without these constraints, the time transformation tends to optimize the points $t(t_i)$ such that constraints can be fulfilled easily.

Joining the function $v_{\mathbb{G}_{N,n_\omega}}$ from (2.2) and any w satisfying the conditions (2.3) and (2.5), yields a feasible discrete control $v(t) \in \mathcal{V}$ defined by

$$v(t) := v_{\mathbb{G}_{N,n_\omega}}(\tau(t)), \qquad t \in [t_0, t_{\mathrm{f}}]$$

see Figure 2.1. Notice, that minor intervals with $w(\tau) = 0$ do not contribute to $v(t)$.

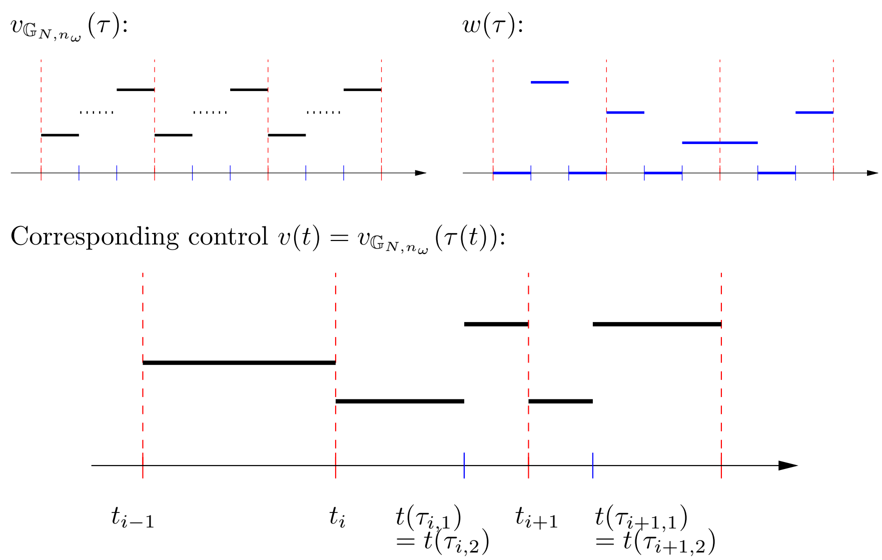

FIG. 2.1. *Back-transformation v (bottom) of variable time transformation for given w and fixed $v_{\mathbb{G}_{N,n_\omega}}$ (top).*

Vice versa, every piecewise constant discrete control v on the major grid $\mathbb{G}_N$ can be described by $v_{\mathbb{G}_{N,n_\omega}}$ and some feasible w.

The preference of values given by the definition of the fixed function $v_{\mathbb{G}_{N,n_\omega}}$ on the minor grid $\mathbb{G}_{N,n_\omega}$ is arbitrary and any other order would be feasible as well. This is not essential, as any discrete control with finitely many jumps can be approximated arbitrarily close on the major grid for h sufficiently small.

Summarizing, the time transformation leads to the following partly discretized optimal control problem:

PROBLEM 2.2.

Minimize

$$\varphi(x(t_0), x(t_{\mathrm{f}}))$$

with respect to $x \in W_{1,\infty}^{n_x}(\mathcal{I})$, $y \in L_\infty^{n_y}(\mathcal{I})$, $u \in L_\infty^{n_u}(\mathcal{I})$, $w \in L_\infty(\mathcal{I})$ subject to

$$\begin{aligned}
\dot{x}(\tau) &= w(\tau) f(x(\tau), y(\tau), u(\tau), v_{\mathbb{G}_{N,n_\omega}}(\tau)) && \text{a.e. in } \mathcal{I}, \\
0_{\mathbb{R}^{n_y}} &= g(x(\tau), y(\tau), u(\tau), v_{\mathbb{G}_{N,n_\omega}}(\tau)) && \text{a.e. in } \mathcal{I}, \\
s(x(\tau)) &\leq 0_{\mathbb{R}^{n_s}} && \text{in } \mathcal{I}, \\
\psi(x(t_0), x(t_\mathrm{f})) &= 0_{\mathbb{R}^{n_\psi}}, \\
u(\tau) &\in \mathcal{U} && \text{a.e. in } \mathcal{I}, \\
w &\in \mathcal{W}.
\end{aligned}$$

Herein, $\mathcal{W}$ is defined by

$$\mathcal{W} := \left\{ w \in L_\infty(\mathcal{I}) \;\middle|\; \begin{array}{l} w(\tau) \geq 0, \\ w \text{ piecewise constant on } \mathbb{G}_{N,n_\omega}, \\ \displaystyle\int_{t_i}^{t_{i+1}} w(\tau) d\tau = t_{i+1} - t_i, \ i = 0, \dots, N \end{array} \right\}.$$

■

Problem 2.2 has only continuous-valued controls and can be solved by a direct discretization method. Application of the inverse time transformation

$$x(t) := \hat{x}(\tau(t)),\ y(t) := \hat{y}(\tau(t)),\ u(t) := \hat{u}(\tau(t)),\ v(t) := v_{G_{N,n_\omega}}(\tau(t))$$

with $\tau(t)$ according to (2.4) to an optimal solution $(\hat{x}, \hat{y}, \hat{u}, \hat{w})$ of Problem 2.2 yields an approximate solution of Problem 2.1.

3. Necessary conditions for optimality. We exploit the time transformation in Section 2 in order to prove a global minimum principle. To this end we consider the following autonomous optimal control problem on a fixed time interval $[t_0, t_f]$ subject to an index one DAE with general set constraints for the control:

Problem 3.1 (Optimal Control Problem). *Let $\mathcal{I} := [t_0, t_f]$ be a non-empty compact time interval with $t_0 < t_f$ fixed. Let*

$$\begin{aligned}
\varphi &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R}, \\
f_0 &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}, \\
f &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_x}, \\
g &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_y}, \\
\psi &: \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R}^{n_\psi}
\end{aligned}$$

be sufficiently smooth functions and $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ a non-empty set.

Minimize the objective function

$$\varphi(x(t_0), x(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} f_0(x(t), y(t), u(t)) \mathrm{d}t$$

with respect to $x \in W_{1,\infty}^{n_x}(\mathcal{I})$, $y \in L_\infty^{n_y}(\mathcal{I})$, $u \in L_\infty^{n_u}(\mathcal{I})$, subject to the DAE

$$\begin{aligned}
\dot{x}(t) &= f(x(t), y(t), u(t)) \quad \text{a.e. in } \mathcal{I}, \\
0_{\mathbb{R}^{n_y}} &= g(x(t), y(t), u(t)) \quad \text{a.e. in } \mathcal{I},
\end{aligned}$$

the boundary condition

$$\psi(x(t_0), x(t_\mathrm{f})) = 0_{\mathbb{R}^{n_\psi}},$$

and the set constraint

$$u(t) \in \mathcal{U} \quad \text{a.e. in } \mathcal{I}.$$

■

The set $\mathcal{U}$ in Problem 3.1 is supposed to be an arbitrary set. We particularly allow that $\mathcal{U}$ may only contain finitely many vectors so that Problem 3.1 contains problems with discrete controls.

In proving necessary conditions we cannot exploit a special structure of $\mathcal{U}$ as it was done in the local minimum principles in [12, 10, p. 95] by assuming that the set was convex with non-empty interior. Hence, these necessary optimality conditions do not hold for Problem 3.1, but they are not worthless. A special time transformation similar to the one in Section 2 is used to transform Problem 3.1 into an equivalent problem with a nice convex control set with non-empty interior for which the local minimum principles are valid. This time transformation is due to Dubovitskii and Milyutin and the following proof techniques in the case of ODEs can be found in [16, p. 148] and [14, p. 95]. The results are extended to the DAE setting in Problem 3.1. To this end let

$$\mathcal{H}(x, y, u, \lambda_f, \lambda_g, \ell_0) := \ell_0 f_0(x, y, u) + \lambda_f^\top f(x, y, u) + \lambda_g^\top g(x, y, u)$$

denote the Hamilton function (also called Hamiltonian) for Problem 3.1, let

$$\tilde{\mathcal{H}}(x, y, u, \lambda_f, \ell_0) := \ell_0 f_0(x, y, u) + \lambda_f^\top f(x, y, u)$$

denote the reduced Hamilton function, and let $(\hat{x}, \hat{y}, \hat{u})$ be a solution of Problem 3.1. Moreover, let the DAE have index one:

Assumption 3.2. *Let the matrix g_y' be uniformly non-singular in the solution $(\hat{x}, \hat{y}, \hat{u})$ and let the inverse matrix $\left(g_y'\right)^{-1}$ be essentially bounded.* ■

As in Section 2 we use the time transformation

$$t(\tau) := t_0 + \int_0^\tau w(r)dr, \quad t(0) = t_0, \quad t(1) = t_\mathrm{f}, \quad w(\tau) \geq 0, \tag{3.1}$$

for $\tau \in [0, 1]$. For any function $w \in L_\infty([0, 1])$ satisfying (3.1) define

$$\begin{aligned}
\tilde{u}(\tau) &:= \begin{cases} \hat{u}(t(\tau)), & \text{for } \tau \in \Delta_w, \\ \text{arbitrary}, & \text{for } \tau \in [0,1] \setminus \Delta_w, \end{cases} \\
\tilde{x}(\tau) &:= \hat{x}(t(\tau)), \\
\tilde{y}(\tau) &:= \begin{cases} \hat{y}(t(\tau)), & \text{for } \tau \in \Delta_w, \\ \text{suitable}, & \text{for } \tau \in [0,1] \setminus \Delta_w, \end{cases}
\end{aligned}$$

where

$$\Delta_w := \{\tau \in [0, 1] \mid w(\tau) > 0\}.$$

Suitable values for $\tilde{y}$ on $[0, 1] \setminus \Delta_w$ will be provided later.

The functions $\tilde{x}$, $\tilde{y}$, $\tilde{u}$ are feasible for the following auxiliary DAE optimal control problem in which w is considered a *control* and $\tilde{u}$ a *fixed* function:

Problem 3.3 (Auxiliary DAE Optimal Control Problem).

Minimize

$$\varphi(x(0), x(1)) + \int_0^1 w(\tau) f_0(x(\tau), y(\tau), \tilde{u}(\tau))d\tau$$

with respect to $x \in W^{n_x}_{1,\infty}([0,1])$, $y \in L^{n_y}_{\infty}([0,1])$, $t \in W_{1,\infty}([0,1])$, $w \in L_{\infty}([0,1])$ subject to the constraints

$$
\begin{aligned}
\dot{x}(\tau) &= w(\tau) f(x(\tau), y(\tau), \tilde{u}(\tau)) && \text{a.e. in } [0,1],\\
0_{\mathbb{R}^{n_y}} &= g(x(\tau), y(\tau), \tilde{u}(\tau)) && \text{a.e. in } [0,1],\\
\dot{t}(\tau) &= w(\tau) && \text{a.e. in } [0,1],\\
0_{\mathbb{R}^{n_\psi}} &= \psi(x(0), x(1)),\\
t(0) &= t_0,\\
t(1) &= t_{\mathrm{f}},\\
w(\tau) &\geq 0 && \text{a.e. in } [0,1]!
\end{aligned}
$$

■

Note that the control w in Problem 3.3 is only restricted by the control constraint $w(\tau) \geq 0$. Consequently, the necessary optimality conditions in [10] hold for Problem 3.3. A formal proof similarly to [16, pp. 149-156] shows that $\tilde{x}$ and $\tilde{y}$ are actually optimal for Problem 3.3 for any feasible control w.

Remark 3.4.

(a) Note: If $w \equiv 0$ on some interval, then on this interval

$$
\begin{aligned}
\dot{x}(\tau) \equiv 0 \quad &\Longrightarrow \quad x(\tau) \equiv \textit{const},\\
\dot{t}(\tau) \equiv 0 \quad &\Longrightarrow \quad t(\tau) \equiv \textit{const}.
\end{aligned}
$$

(b) The necessary conditions in [10] require the functions

$$
\begin{aligned}
\tilde{f}_0(\tau, x, y, w) &:= w f_0(x, y, \tilde{u}(\tau)),\\
\tilde{f}(\tau, x, y, w) &:= w f(x, y, \tilde{u}(\tau)),\\
\tilde{g}(\tau, x, y) &:= g(x, y, \tilde{u}(\tau))
\end{aligned}
$$

to be continuous with respect to the component τ and continuously differentiable with respect to x, y, and w. This assumption is not satisfied in general for Problem 3.3 as $\tilde{u}$ is not continuous in general, but it can be relaxed appropriately for measurable w.r.t. τ functions.

■

First order necessary optimality conditions for Problem 3.3 with the augmented Hamilton function

$$
\begin{aligned}
&\bar{\mathcal{H}}(\tau, x, y, t, w, \lambda_f, \lambda_g, \lambda_t, \eta, \ell_0)\\
&\quad := w\left(\ell_0 f_0(x, y, \tilde{u}(\tau)) + \lambda_f^\top f(x, y, \tilde{u}(\tau)) + \lambda_t - \eta\right) + \lambda_g^\top g(x, y, \tilde{u}(\tau))\\
&\quad = w\left(\tilde{\mathcal{H}}(x, y, \tilde{u}(\tau), \lambda_f, \ell_0) + \lambda_t - \eta\right) + \lambda_g^\top g(x, y, \tilde{u}(\tau))
\end{aligned}
$$

read as follows: There exist multipliers $\tilde{\ell}_0 \in \mathbb{R}$, $\tilde{\lambda}_f \in W^{n_x}_{1,\infty}([0,1])$, $\tilde{\lambda}_g \in L^{n_y}_{\infty}([0,1])$, $\tilde{\lambda}_t \in W_{1,\infty}([0,1])$, $\tilde{\eta} \in L_{\infty}([0,1])$, and $\tilde{\sigma} \in \mathbb{R}^{n_\psi}$, not all zero, with

(a) $\tilde{\ell}_0 \geq 0$

(b) In $[0,1]$ we have the adjoint equation

$$
\begin{aligned}
\frac{d}{d\tau}\tilde{\lambda}_f(\tau) &= -w(\tau)\tilde{\mathcal{H}}'_x(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau), \tilde{\lambda}_f(\tau), \tilde{\ell}_0)^\top - g'_x(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau))^\top \tilde{\lambda}_g(\tau),\\
0_{\mathbb{R}^{n_y}} &= \quad w(\tau)\tilde{\mathcal{H}}'_y(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau), \tilde{\lambda}_f(\tau), \tilde{\ell}_0)^\top + g'_y(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau))^\top \tilde{\lambda}_g(\tau),\\
\frac{d}{d\tau}\tilde{\lambda}_t(\tau) &= 0.
\end{aligned}
$$

In particular, $\tilde{\lambda}_t$ is constant.

(c) Transversality conditions:

$$\tilde{\lambda}_f(0)^\top = -\left(\tilde{\ell}_0\varphi'_{x_0} + \tilde{\sigma}^\top\psi'_{x_0}\right), \quad \tilde{\lambda}_f(1)^\top = \tilde{\ell}_0\varphi'_{x_f} + \tilde{\sigma}^\top\psi'_{x_f}.$$

(d) Almost everywhere in $[0,1]$ it holds

$$0 = \tilde{\mathcal{H}}(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau), \tilde{\lambda}_f(\tau), \tilde{\ell}_0) + \tilde{\lambda}_t(\tau) - \tilde{\eta}(\tau).$$

Owing to the complementarity condition in (e) we thus have

$$\tilde{\mathcal{H}}(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau), \tilde{\lambda}_f(\tau), \tilde{\ell}_0) + \tilde{\lambda}_t(\tau) \begin{cases} = 0, & \text{if } \tau \in \Delta_w, \\ \geq 0, & \text{if } \tau \notin \Delta_w. \end{cases}$$

(e) Almost everywhere in $[0,1]$ it holds

$$\tilde{\eta}(\tau)w(\tau) = 0 \quad \text{and} \quad \tilde{\eta}(t) \geq 0.$$

Using the inverse time transformation defined in (2.4) we may define $\ell_0 := \tilde{\ell}_0$, $\sigma := \tilde{\sigma}$, $\lambda_f(t) := \tilde{\lambda}_f(\tau(t))$, $\lambda_t(t) := \tilde{\lambda}_t(\tau(t))$, and in addition

$$\lambda_g(t)^\top := -\tilde{\mathcal{H}}'_y(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \ell_0)\left(g'_y(\hat{x}(t), \hat{y}(t), \hat{u}(t))\right)^{-1}.$$

Then, for almost every $\tau \in \Delta_w$ it holds

$$\lambda_f(t(\tau)) = \tilde{\lambda}_f(\tau), \quad \lambda_t(t(\tau)) = \tilde{\lambda}_t(\tau),$$

and λ_f and λ_g satisfy the adjoint equation

$$\begin{aligned} \dot{\lambda}_f(t) &= -\mathcal{H}'_x(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \lambda_g(t), \ell_0)^\top \\ 0_{\mathbb{R}^{n_y}} &= \quad \mathcal{H}'_y(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \lambda_g(t), \ell_0)^\top, \end{aligned}$$

and the transversality conditions

$$\lambda_f(t_0)^\top = -\left(\ell_0\varphi'_{x_0} + \sigma^\top\psi'_{x_0}\right), \quad \lambda_f(t_\mathrm{f})^\top = \ell_0\varphi'_{x_f} + \sigma^\top\psi'_{x_f}.$$

The above conditions hold for every w. Now, we will choose w in a special way in order to exploit this degree of freedom. The following construction follows [16, p. 157], compare Figure 3.1, and was exploited numerically in Section 2.

Let $w(\tau)$ vanish on the intervals $\mathcal{I}_k := (\tau_k, \tau_k + \beta_k]$, $k = 1, 2, \ldots$, which are to be constructed such that the image of $\bigcup_k \mathcal{I}_k$ under the mapping $\tau \mapsto t(\tau)$ is dense in $\mathcal{I}$. To this end let $\{\xi_1, \xi_2, \ldots\}$ be a countable dense subset of $\mathcal{I}$. Choose $\beta_k > 0$ with $\sum_k \beta_k = \frac{1}{2}$ and let

$$\tau_k := \frac{\xi_k - t_0}{2(t_\mathrm{f} - t_0)} + \sum_{j:\xi_j < \xi_k} \beta_j.$$

Then, the intervals $\mathcal{I}_k = (\tau_k, \tau_k + \beta_k]$ are pairwise disjoint. Define

$$w(\tau) := \begin{cases} 0, & \text{if } \tau \in \bigcup_k \mathcal{I}_k, \\ 2(t_\mathrm{f} - t_0), & \text{if } \tau \notin \bigcup_k \mathcal{I}_k. \end{cases}$$

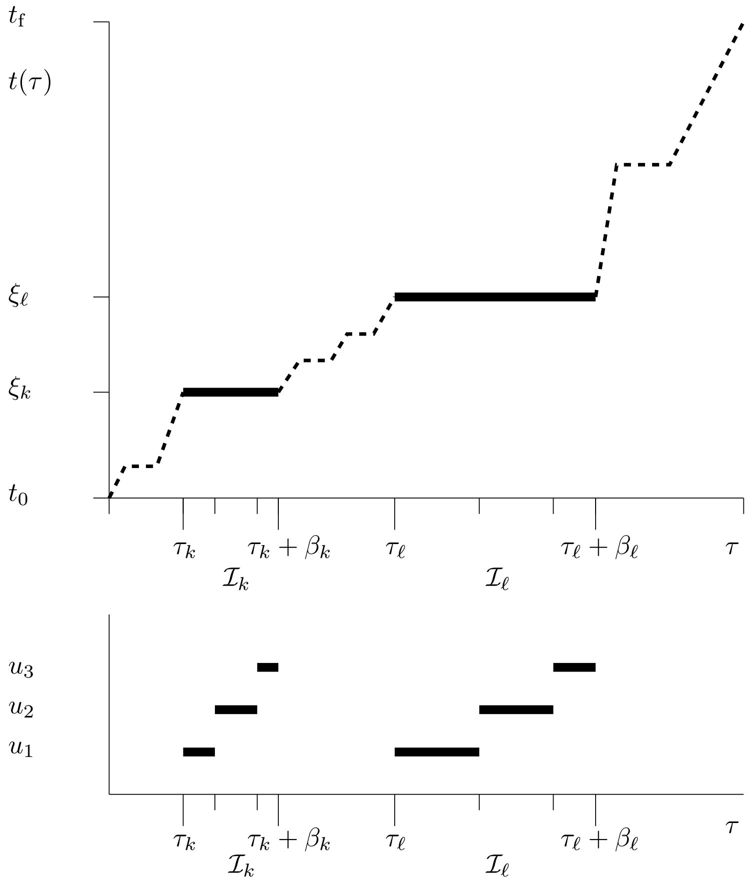

FIG. 3.1. *Construction principle of w.*

We will show that $t(\tau) = \xi_k$ for any $\tau \in \mathcal{I}_k$. As $\{\xi_k\}$ was chosen to be dense in $\mathcal{I}$, so is the image of $\bigcup_k \mathcal{I}_k$ under the mapping $\tau \mapsto t(\tau)$.

We note that $\tau_j < \tau_k$ if and only if $\xi_j < \xi_k$ and $t(\tau) = t(\tau_k)$ for all $\tau \in \mathcal{I}_k$. For $\tau \in \mathcal{I}_k$ we find

$$\begin{aligned} t(\tau) &= t_0 + \int_0^\tau w(\xi)d\xi \\ &= t_0 + 2(t_{\mathrm{f}} - t_0)\left(\tau_k - \sum_{j:\tau_j<\tau_k} \beta_j\right) \\ &= t_0 + 2(t_{\mathrm{f}} - t_0)\left(\tau_k - \sum_{j:\xi_j<\xi_k} \beta_j\right) \\ &= t_0 + (t_{\mathrm{f}} - t_0)\frac{\xi_k - t_0}{t_{\mathrm{f}} - t_0} = \xi_k. \end{aligned}$$

Now let

(i) $\mathcal{I}_k = \bigcup_j \mathcal{I}_{kj}$, where $\mathcal{I}_{kj}$ are nonempty closed from the right intervals;
(ii) $\{u_1, u_2, \ldots\}$ be a countable dense subset of $\mathcal{U}$;
(iii) $\tilde{u}(\tau) := u_j$ if $\tau \in \mathcal{I}_{kj}$;

(iv) $\tilde{y}(\tau) := y_j$ with $0_{\mathbb{R}^{n_y}} = g(\tilde{x}(\tau), y_j, u_j)$ if $\tau \in \mathcal{I}_{kj}$;
According to (d) we have for almost every $\tau \in \bigcup_k \mathcal{I}_k$ the inequality

$$\tilde{\mathcal{H}}(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau), \tilde{\lambda}_f(\tau), \tilde{\ell}_0) + \tilde{\lambda}_t(\tau) \geq 0.$$

As every interval $\mathcal{I}_{kj}$ has a positive measure, there exists $\tau \in \mathcal{I}_{kj}$ with $t(\tau) = \xi_k$ such that

$$\begin{aligned}&\tilde{\mathcal{H}}(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau), \tilde{\lambda}_f(\tau), \tilde{\ell}_0) + \tilde{\lambda}_t(\tau)\\&\quad = \tilde{\mathcal{H}}(\hat{x}(\xi_k), y_j, u_j, \lambda_f(\xi_k), \ell_0) + \lambda_t(\xi_k) \geq 0.\end{aligned}$$

Since the set $\{\xi_1, \xi_2, \ldots\}$ is dense in $\mathcal{I}$, $\{u_1, u_2, \ldots\}$ is dense in $\mathcal{U}$, and

$$h(t, y, u) := \tilde{\mathcal{H}}(\hat{x}(t), y, u, \lambda_f(t), \ell_0)$$

is continuous, it follows for almost all $t \in \mathcal{I}$ that

$$\tilde{\mathcal{H}}(\hat{x}(t), y, u, \lambda_f(t), \ell_0) + \lambda_t(t) \geq 0$$

for all

$$(u, y) \in M(\hat{x}(t)) := \{(u, y) \in \mathcal{U} \times \mathbb{R}^{n_y} \mid 0_{\mathbb{R}^{n_y}} = g(\hat{x}(t), y, u)\}.$$

Note, that $\mathcal{H} \equiv \tilde{\mathcal{H}}$ whenever $(u, y) \in M(\hat{x}(t))$.

On the other hand, according to (d) for almost every $\tau \in \Delta_w$ and thus for almost every $t \in \mathcal{I}$ it holds

$$\tilde{\mathcal{H}}(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \ell_0) + \lambda_t(t) = 0.$$

Putting both relations together yields in the minimality of the reduced Hamilton function for almost every $t \in \mathcal{I}$:

$$\begin{aligned}&\tilde{\mathcal{H}}(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \ell_0)\\&\quad \leq \tilde{\mathcal{H}}(\hat{x}(t), y, u, \lambda_f(t), \ell_0) \quad \text{for all } (u, y) \in M(\hat{x}(t)).\end{aligned}$$

Moreover, since $\hat{u}$ is essentially bounded and $h(t, y, u)$ is continuous, it follows

$$\tilde{\mathcal{H}}(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \ell_0) + \lambda_t(t) \equiv 0$$

almost everywhere. Since λ_t is continuous and constant according to (b), so is $\tilde{\mathcal{H}}$ as a function of time. Exploiting $\tilde{\mathcal{H}} \equiv \mathcal{H}$ for every $(u, y) \in M(\hat{x}(t))$ we have thus proved the following global minimum principle:

THEOREM 3.5 (Global Minimum Principle). *Let the following assumptions be fulfilled for the optimal control problem 3.1.*

(i) Let the functions φ, f_0, f, g, ψ be continuous with respect to all arguments and continuously differentiable with respect to x and y.
(ii) Let $(\hat{x}, \hat{y}, \hat{u})$ be a strong local minimum of the optimal control problem 3.1.
(iii) Let Assumption 3.2 be valid.

Then there exist multipliers $\ell_0 \in \mathbb{R}$, $\lambda_f \in W^{n_x}_{1,\infty}(\mathcal{I})$, $\lambda_g \in L^{n_y}_{\infty}(\mathcal{I})$, $\sigma \in \mathbb{R}^{n_\psi}$ such that the following conditions are satisfied:

(a) $\ell_0 \geq 0$, $(\ell_0, \sigma, \lambda_f, \lambda_g) \neq \Theta$,

(b) Adjoint equations*: Almost everywhere in $\mathcal{I}$ it holds*

$$\begin{aligned}\dot{\lambda}_f(t) &= -\mathcal{H}'_x(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \lambda_g(t), \ell_0)^\top, \\ 0_{\mathbb{R}^{n_y}} &= \mathcal{H}'_y(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \lambda_g(t), \ell_0)^\top.\end{aligned}$$

(c) Transversality conditions*:*

$$\begin{aligned}\lambda_f(t_0)^\top &= -\left(\ell_0 \varphi'_{x_0}(\hat{x}(t_0), \hat{x}(t_f)) + \sigma^\top \psi'_{x_0}(\hat{x}(t_0), \hat{x}(t_f))\right), \\ \lambda_f(t_f)^\top &= \ell_0 \varphi'_{x_f}(\hat{x}(t_0), \hat{x}(t_f)) + \sigma^\top \psi'_{x_f}(\hat{x}(t_0), \hat{x}(t_f)).\end{aligned}$$

(d) Optimality condition*: Almost everywhere in $\mathcal{I}$ it holds*

$$\mathcal{H}(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \lambda_g(t), \ell_0) \le \mathcal{H}(\hat{x}(t), y, u, \lambda_f(t), \lambda_g(t), \ell_0)$$

for all $(u, y) \in M(\hat{x}(t))$, where

$$M(x) := \{(u, y) \in \mathcal{U} \times \mathbb{R}^{n_y} \mid g(x, y, u) = 0_{\mathbb{R}^{n_y}}\}.$$

(e) The Hamilton function is constant w.r.t. time:

$$\mathcal{H}(\hat{x}(t), \hat{y}(t), \hat{u}(t), \lambda_f(t), \lambda_g(t), \ell_0) \equiv \mathit{const.}$$

■

Herein, $(\hat{x}, \hat{y}, \hat{u})$ is called *strong local minimum*, if $(\hat{x}, \hat{y}, \hat{u})$ minimizes the objective function among all feasible functions with $\|x - \hat{x}\|_\infty < \varepsilon$ for some $\varepsilon > 0$.

Example 3.6 ([26, p. 620]). *Consider the following optimal control problem:*

Minimize

$$\int_0^1 x(t)^2 + \alpha(y(t) - u(t))^2 \mathrm{d}t$$

subject to the constraints

$$\begin{aligned}\dot{x}(t) &= y(t) - u(t), \quad x(0) = 0, \\ 0 &= y(t) - u(t), \\ u(t) &\in \mathcal{U} := [-1, 1].\end{aligned}$$

Apparently, every *feasible control is optimal!*
Hamilton function:

$$\mathcal{H}(x, y, u, \lambda_f, \lambda_g, \ell_0) = \ell_0 x^2 + \alpha(y - u)^2 + \lambda_f(y - u) + \lambda_g(y - u)$$

Minimization of $\mathcal{H}$ with respect to

$$(u, y) \in M = \{(u, y) \in \mathcal{U} \times \mathbb{R} \mid y - u = 0\}$$

yields that every u satisfies the global minimum principle.

Please note that the simultaneous coupling between u and y by means of the set M is important in the global minimum principle. A wrong condition would be obtained, if the Hamilton function was firstly minimized with respect to u (assuming y to be fixed) and the consistent algebraic variable y corresponding to the minimizing u was *determined afterwards. For instance consider the case $\alpha = -1$ and $\hat{u} \equiv 0$. The algebraic equation, the adjoint equation and the transversality condition yield*

$$\dot{\lambda}_f(t) = -\ell_0 \hat{x}(t) = 0, \quad \lambda_f(1) = 0 \quad \Longrightarrow \quad \lambda_f(t) \equiv 0.$$

Moreover, $g(\hat{x}(t), \hat{y}(t), \hat{u}(t)) \equiv 0$ and hence

$$\mathcal{H}(\hat{x}, \hat{y}, \hat{u}, \lambda_f, \lambda_g, \ell_0) = \alpha(\hat{y} - \hat{u})^2 = -(\hat{y} - \hat{u})^2.$$

Minimizing $\mathcal{H}$ with respect to $u \in [-1, 1]$ yields either $\hat{u} = +1$ or $\hat{u} = -1$ depending on $\hat{y} \in [-1, 1]$. This contradicts $\hat{u} \equiv 0$. ∎

The global minimum principle allows to prove additional properties of the Hamilton function in the case of a free final time. In this case, the Hamilton function vanishes almost everywhere.

THEOREM 3.7. *Let the assumptions of Theorem 3.5 hold and let the final time in Problem 3.1 be free. Then, $\mathcal{H}$ vanishes almost everywhere and*

$$\mathcal{H}(\hat{x}(t_f), \hat{y}(t_f), \hat{u}(t_f), \lambda_f(t_f), \lambda_g(t_f), \ell_0) = 0.$$

Proof. We use standard transformation techniques to transform the problem to an equivalent problem on a fixed time interval:

Minimize

$$\varphi(\bar{x}(0), \bar{x}(1)) + \int_0^1 (t_f(\tau) - t_0) f_0(\bar{x}(\tau), \bar{y}(\tau), \bar{u}(\tau)) d\tau$$

subject to the constraints

$$\begin{aligned}
\frac{d}{d\tau}\bar{x}(\tau) &= (t_f(\tau) - t_0) f(\bar{x}(\tau), \bar{y}(\tau), \bar{u}(\tau)) && \text{a.e. in } [0,1],\\
0_{\mathbb{R}^{n_y}} &= (t_f(\tau) - t_0) g(\bar{x}(\tau), \bar{y}(\tau), \bar{u}(\tau)) && \text{a.e. in } [0,1],\\
\frac{d}{d\tau} t_f(\tau) &= 0 && \text{in } [0,1],\\
\psi(\bar{x}(0), \bar{x}(1)) &= 0_{n_\psi}, &&\\
\bar{u}(\tau) &\in \mathcal{U} && \text{a.e. in } [0,1].
\end{aligned}$$

The Hamilton function for the transformed problem reads as

$$\bar{\mathcal{H}}(\bar{x}, \bar{y}, t_{\mathrm{f}}, \bar{u}, \bar{\lambda}_f, \bar{\lambda}_g, \bar{\lambda}_t, \bar{\ell}_0) = (t_{\mathrm{f}} - t_0)\mathcal{H}(\bar{x}, \bar{y}, \bar{u}, \bar{\lambda}_f, \bar{\lambda}_g, \bar{\ell}_0),$$

where $\mathcal{H}$ denotes the Hamilton function of the original problem. Theorem 3.5 yields the following adjoint equation and transversality conditions for the adjoint $\bar{\lambda}_t$:

$$\frac{d}{d\tau}\bar{\lambda}_t(\tau) = -\bar{\mathcal{H}}'_{t_{\mathrm{f}}}[\tau] = -\mathcal{H}[\tau], \quad \bar{\lambda}_t(0) = \bar{\lambda}_t(1) = 0. \tag{3.2}$$

According to part (e) of Theorem 3.5 the Hamilton function $\bar{\mathcal{H}}$ is constant almost everywhere and thus $\mathcal{H}$ is constant almost everywhere as well since t_{f} is constant. From (3.2) it follows $\mathcal{H}[\tau] = 0$ almost everywhere and particularly after back-transformation $\mathcal{H}(\hat{x}(t_{\mathrm{f}}), \hat{y}(t_{\mathrm{f}}), \hat{u}(t_{\mathrm{f}}), \lambda_f(t_{\mathrm{f}}), \lambda_g(t_{\mathrm{f}}), \ell_0) = 0$. ∎

4. On the relation between relaxed and integer solutions. The necessary conditions of optimality that have been derived in the previous section are important, as they can be used from an algorithmical point of view and to gain analytical insight in solution structures. In practice, however, often *first discretize, then optimize* methods are used, compare [29] for an overview. Whenever integer programming is applied, lower bounds obtained from relaxations are crucial. In this context a *partial outer convexification* has been proposed for mixed–integer optimal control problems in ordinary differential equations, [28]. We are going to extend this to the case of MIOCP in DAE as formulated in Problem 2.1.

PROBLEM 4.1 (MIOCP after Outer Convexification). *Let $\mathcal{I}, \varphi, f, g, s, \psi, \mathcal{U}, \mathcal{V}$ be defined as in Problem 2.1.*

Let the matrix $g'_y(x, y, u, v^i)$ be uniformly non-singular for all arguments and let the inverse matrix $\left(g'_y(x, y, u, v^i)\right)^{-1}$ be essentially bounded for all $i = 1 \dots n_\omega$.

Minimize the objective function

$$\varphi(x(t_0), x(t_{\mathrm{f}}))$$

with respect to $x \in W^{n_x}_{1,\infty}(\mathcal{I})$, $u \in L^{n_u}_\infty(\mathcal{I})$, $\omega \in L^{n_\omega}_\infty(\mathcal{I})$ subject to the system of ODEs

$$\dot{x}(t) = \sum_{i=1}^{n_\omega} \omega_i(t)\ f\left(x(t), \theta^{i,t}\left(x(t), u(t)\right), u(t), v^i\right) \quad \text{a.e. in } \mathcal{I},$$

with continuous mappings $\theta^{i,t} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_y}$ for $i = 1 \dots n_\omega$ and $t \in \mathcal{I}$ a.e., to the state constraint

$$s(x(t)) \leq 0_{\mathbb{R}^{n_s}},$$

the boundary conditions

$$\psi(x(t_0), x(t_{\mathrm{f}})) = 0_{\mathbb{R}^{n_\psi}},$$

the set constraints

$$\begin{aligned} u(t) &\in \mathcal{U} && \text{a.e. in } \mathcal{I}, \\ \omega(t) &\in \{0,1\}^{n_\omega} && \text{a.e. in } \mathcal{I}, \end{aligned}$$

and the special ordered set type 1 condition

$$\sum_{i=1}^{n_\omega} \omega_i(t) = 1 \quad \text{a.e. in } \mathcal{I}. \tag{4.1}$$

■

THEOREM 4.2. *Let $(x^*, y^*, u^*, v^*)(\cdot)$ be an optimal solution of Problem 2.1. Define $\omega^* : \mathcal{I} \longrightarrow \{0,1\}^{n_\omega}$ as*

$$\omega_i^*(t) := \begin{cases} 1 & \textit{if } v^*(t) = v^i \\ 0 & \textit{else} \end{cases}$$

Then continuous mappings $\theta^{i,t} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_y}$ exist such that

$$y^*(t) = \sum_{i=1}^{n_\omega} \omega_i^*(t)\ \theta^{i,t}\left(x^*(t), u^*(t)\right) \quad \textit{a.e. in } \mathcal{I} \tag{4.2}$$

and $(x^*, u^*, \omega^*)(\cdot)$ *is an optimal solution of Problem 4.1.*

If, conversely, $(x^*, u^*, \omega^*)(\cdot)$ *is an optimal solution of Problem 4.1 for mappings* $\theta^{i,t}(\cdot)$ *that fulfill*

$$0_{\mathbb{R}^{n_y}} = \sum_{i=1}^{n_\omega} \omega_i^*(t)\; g\left(x^*(t), \theta^{i,t}\left(x^*(t), u^*(t)\right), u^*(t), v^i\right) \tag{4.3}$$

a.e. in $\mathcal{I}$ *and we define*

$$v^*(t) = v^i \text{ if and only if } \omega_i^*(t) = 1, \tag{4.4}$$

then also $(x^*, y^*, u^*, v^*)(\cdot)$ *with* $y^*(\cdot)$ *given by* (4.2) *is an optimal solution of Problem 2.1.*

Proof. Assume (x^*, y^*, u^*, v^*) to be a feasible solution of Problem 2.1. By construction it holds

$$v^*(t) = \sum_{i=1}^{n_\omega} \omega_i^*(t)\; v^i \tag{4.5}$$

for a.e. t in $\mathcal{I}$. As (x^*, y^*, u^*, v^*) is feasible, in particular

$$0_{\mathbb{R}^{n_y}} = g(x^*(t), y^*(t), u^*(t), v^*(t))$$

holds. Because of the index one assumption the matrices

$$\frac{\partial g(x^*(t), y^*(t), u^*(t), v^i)}{\partial y}$$

are invertible for all $i = 1 \dots n_\omega$. Hence the implicit function theorem states for all $i = 1 \dots n_\omega$ and $t \in \mathcal{I}$ a.e. the existence of open sets $\mathcal{S}^{1,i,t} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ and $\mathcal{S}^{2,i,t} \subset \mathbb{R}^{n_y}$ with $s^{1,i,*}(t) := (x^*, u^*)(t) \in \mathcal{S}^{1,i,t}$ and $s^{2,i,*}(t) := y^*(t) \in \mathcal{S}^{2,i,t}$ and continuous mappings $\theta^{i,t} : \mathcal{S}^{1,i,t} \longrightarrow \mathcal{S}^{2,i,t}$ exist such that

$$0_{\mathbb{R}^{n_y}} = g(s^1, \theta^{i,t}(s^1), v^i) \tag{4.6}$$

for all $(s^1, s^2) \in \mathcal{S}^{1,i,t} \times \mathcal{S}^{2,i,t}$ with a slight abuse of notation in the order of the arguments. In particular, it holds (4.3) for $s^1 = (x^*, u^*)(t)$.

Hence it is possible to pointwise replace $y^*(t)$ in Problem 2.1 by

$$y^*(t) = \sum_{i=1}^{n_\omega} \omega_i^*(t)\; \theta^{i,t}\left(x^*(t), u^*(t)\right). \tag{4.7}$$

Substituting (4.5) and (4.7) into the right hand side function $f(\cdot)$ yields equivalence,

$$\begin{aligned} \dot{x}(t) &= f(x^*(t), y^*(t), u^*(t), v^*(t)) \\ &= f\left(x^*(t), \sum_{i=1}^{n_\omega} \omega_i^*(t)\; \theta^{i,t}\left(x^*(t), u^*(t)\right), u^*(t), \sum_{i=1}^{n_\omega} \omega_i^*(t)\; v^i\right) \\ &= \sum_{i=1}^{n_\omega} \omega_i^*(t)\; f\left(x^*(t), \theta^{i,t}\left(x^*(t), u^*(t)\right), u^*(t), v^i\right), \end{aligned}$$

making use of $\omega_i^* \in \{0, 1\}$ and $\sum_{i=1}^{n_\omega} \omega_i^*(t) = 1$. Hence (x^*, u^*, ω^*) is also a feasible solution of Problem 4.1. This solution is also optimal, because any better trajectory

(x^*, u^*, ω^*) can be mapped via (4.4) and (4.7) to a feasible trajectory of Problem 2.1 with the same objective function value, contradicting the original optimality assumption.

Let us now assume (x^*, u^*, ω^*) to be an optimal solution of Problem 4.1 for mappings $\theta^{i,t}(\cdot)$ that fulfill (4.3) a.e. in $\mathcal{I}$. We define $v^*(\cdot)$ via (4.4) and $y^*(\cdot)$ via (4.7) and again have an equivalence of the right hand sides of the differential equations, which results in identical feasibility and optimality. ∎

Note that the proof is similar to the one in [28] for the ODE case. The main extension is the equivalence of the ODE formulation to the DAE in Problem 2.1. It follows from the index 1 assumption which allows to apply the implicit function theorem.

Theorem 4.2 allows us to transfer formally a MIOCP in DAE in which the discrete controls $v(\cdot)$ enter in an arbitrary manner to a MIOCP in ODE in which the discrete controls $\omega(\cdot)$ enter linearly. This is important, as it allows to apply the integer gap theorem from [31] to Problem 4.1. It states that for any given tolerance ϵ a control discretization grid size can be determined, such that the difference of norms of the optimal trajectory of a relaxed ($\omega_i(t) \in [0, 1]$) and an integer valued ($\omega_i(t) \in \{0, 1\}$) trajectory is smaller than ϵ. This property carries over by continuity arguments to the objective function and state constraints.

COROLLARY 4.3. *If $f(\cdot)$ and $g(\cdot)$ are sufficiently smooth, the optimal objective function value of Problem 2.1 is given by the optimal objective function value of Problem 4.1, where the requirements $\omega_i(t) \in \{0, 1\}$ have been relaxed to $\omega_i(t) \in [0, 1]$.*

The proof is a simple combination of Theorem 4.2 and [31, Corollary 8], in which also precise assumptions on the smoothness of $f(\cdot)$ are formulated.

It also allows to apply the Sum Up Rounding strategy to calculate integer controls from relaxed controls in linear time, compare also [31]. Interestingly, also the Sum Up Rounding strategy is a constructive element of a theoretical proof, similar to the time transformation in the previous sections.

From a practical point of view, however, it is not always desirable or even possible to explicitly calculate the mappings $\theta^{i,t}(\cdot)$. The question how the algebraic equations can be altered to maintain a DAE formulation with a property similar to Corollary 4.3 is an open question, like the extension to DAE systems for which the index one assumption does not hold.

5. Numerical benchmark. We will illustrate the results from the previous sections with a MIOC benchmark example from the literature. Our focus is not on efficient numerical methods for MIOCPs, which have been described, e.g., in [13, 33]. Rather we give an example of a boundary value problem that is deduced directly from the necessary conditions of optimality and follow a *first optimize, then discretize* approach.

We choose a control problem motivated by F8 aircraft control. It is based on ordinary differential equations (ODE), but we reformulate it with the help of two artificial algebraic variables. This is not necessarily helpful from a computational point of view, but will allow us to compare the results to published results for this benchmark problem [30], and we can be certain that for this test problem the index one assumption 3.2 is always valid.

The differential equations were introduced by Garrard [9]. The differential states consist of $x_1(\cdot)$ as the angle of attack in radians, $x_2(\cdot)$ as the pitch angle, and $x_3(\cdot)$ as the pitch rate in rad/s. The F-8 aircraft control problem was introduced by Kaya and

Noakes [17] and aims at controlling an aircraft in a time-optimal way from an initial state to a terminal state. The only control function $v(\cdot)$ is the tail deflection angle in radians which may attain only two different values. For $t \in [0, t_f]$ almost everywhere the mixed-integer optimal control problem is given by

Problem 5.1 (F-8 Aircraft Control with algebraic variables).
Minimize $\int_0^{t_f} 1 \, dt$
with respect to $x \in W^3_{1,\infty}(\mathcal{I})$, $y \in L^2_\infty(\mathcal{I})$, $v \in L^1_\infty(\mathcal{I})$ subject to

$$\begin{aligned}
\dot{x}_1(t) &= y_1(t)x_1(t) + x_3(t) - 0.019x_2(t)^2 - 0.215v(t) + 0.63v(t)^3, && (5.1)\\
\dot{x}_2(t) &= x_3(t), && (5.2)\\
\dot{x}_3(t) &= -4.208x_1(t) - 0.396x_3(t) - 0.47x_1(t)^2 - 3.564x_1(t)^3 && (5.3)\\
&\quad -20.967v(t) + 6.265x_1(t)y_2(t) + 46y_2(t)v(t) + 61.4v(t)^3, && (5.4)\\
0 &= -y_1(t) - 0.877 - 0.088x_3(t) + 0.47x_1(t) - x_1(t)x_3(t) && (5.5)\\
&\quad +3.846x_1(t)^2 + 0.28y_2(t) + 0.47v(t)^2, && (5.6)\\
0 &= -y_2(t) + x_1(t)v(t), && (5.7)\\
v(t) &\in \{-0.05236, 0.05236\}, && (5.8)\\
x(0) &= (0.4655, 0, 0)^\top, \quad x(t_f) = (0, 0, 0)^\top. && (5.9)
\end{aligned}$$

Note that the problem formulation in [17, 30] can be easily regained by substituting first $y_2(\cdot)$ and then $y_1(\cdot)$ back in the differential equations. In the following we leave the time argument (t) away for notational simplicity. We look at the necessary conditions of optimality for the case without time transformation, which we apply in our numerical scheme.

With $\ell_0 = 1$, the Hamiltonian of Problem 5.1 is given by

$$\begin{aligned}
\mathcal{H}(\cdot) &= 1 + \lambda_f^\top f(x, y, u) + \lambda_g^\top g(x, y, u)\\
&= 1 + \lambda_{f1}(y_1x_1 + x_3 - 0.019x_2^2 - 0.215v + 0.63v^3) \qquad (5.10)\\
&\quad +\lambda_{f2}x_3\\
&\quad +\lambda_{f3}(-4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3\\
&\qquad -20.967v + 6.265x_1y_2 + 46y_2v + 61.4v^3)\\
&\quad +\lambda_{g1}(-y_1 - 0.877 - 0.088x_3 + 0.47x_1\\
&\qquad -x_1x_3 + 3.846x_1^2 + 0.28y_2 + 0.47v^2)\\
&\quad +\lambda_{g2}(-y_2 + x_1v)
\end{aligned}$$

The adjoint equations read

$$\begin{aligned}
-\dot{\lambda}_{f1} &= \frac{\partial \mathcal{H}(\cdot)}{\partial x_1} = \lambda_{f3}(-4.208 - 2\cdot 0.47x_1 - 3\cdot 3.564x_1^2 + 6.265y_2) && (5.11)\\
&\qquad +\lambda_{f1}y_1 + \lambda_{g1}(0.47 - x_3 + 2\cdot 3.846x_1) + \lambda_{g2}v\\
-\dot{\lambda}_{f2} &= \frac{\partial \mathcal{H}(\cdot)}{\partial x_2} = -0.038\lambda_{f1}x_2 && (5.12)\\
-\dot{\lambda}_{f3} &= \frac{\partial \mathcal{H}(\cdot)}{\partial x_3} = \lambda_{f1} + \lambda_{f2} - 0.396\lambda_{f3} + \lambda_{g1}(-0.088 - x_1) && (5.13)\\
0 &= \frac{\partial \mathcal{H}(\cdot)}{\partial y_1} = \lambda_{f1}x_1 - \lambda_{g1} && (5.14)\\
0 &= \frac{\partial \mathcal{H}(\cdot)}{\partial y_2} = \lambda_{f3}(6.265x_1 + 46v) + 0.28\lambda_{g1} - \lambda_{g2} && (5.15)
\end{aligned}$$

with the transversality conditions

$$\lambda_{fi}(t_0) = -\frac{\partial\varphi(\cdot) + \sigma^\top \psi(\cdot)}{\partial x_i(t_0)} = \sigma_{0i}, \quad \lambda_{fi}(t_\mathrm{f}) = -\frac{\partial\varphi(\cdot) + \sigma^\top \psi(\cdot)}{\partial x_i(t_\mathrm{f})} = \sigma_{fi}, \tag{5.16}$$

for $i = 1 \dots 3$. In other words, the initial and terminal values of the differential adjoint states are free, because all initial and terminal values of the differential states $x_i(\cdot)$ are fixed. However, the value of the Hamiltonian at the free end time is fixed to 0,

$$\mathcal{H}(\cdot, t_\mathrm{f}) = 0. \tag{5.17}$$

The value of the optimal control $v(t)$ is determined according to the global minimum principle in Theorem 3.5 as the pointwise minimizer

$$v^*(t) = \arg\min\{\mathcal{H}(\cdot, v \equiv -0.05236), \mathcal{H}(\cdot, v \equiv 0.05236)\}. \tag{5.18}$$

Fast and reliable methods for mixed-integer optimal control problems have been developed recently [13, 33]. They are based, among others, on the transformations that have been described in the previous sections. For illustration and validation, however, we solve the boundary problem that results from the global minimum principle at this point. As a numerical solution we apply the method of Competing Hamiltonians that to our knowledge was first proposed in [3]. It enumerates the values of the Hamiltonian for all discrete control choices and chooses the pointwise minimizer, in our case as (5.18). Note that this approach is restricted to optimal control problems that have optimal bang-bang solutions for their relaxations, [33].

The resulting boundary value problem consists of determining $(x^*, y^*, \lambda_f^*, \lambda_g^*, v^*)(\cdot)$ and a final time t_f such that (5.1–5.9, 5.11–5.15, 5.16, 5.17, 5.18) are fulfilled.

We need good initial guesses for the switching structure and for the differential and algebraic variables. We obtain them by running the MS MINTOC algorithm [28, 33]. It is based on a *first discretize, then optimize* approach, in particular on Bock's direct multiple shooting method [4]. We can directly read of the optimal switching structure and the values of differential and algebraic variables. For the adjoint variables $\lambda_f(\cdot)$ we take Lagrange multipliers of the matching conditions as a discrete approximation. This pretty accurate initialization allows us to solve the DAE boundary value problem despite its well known small region of convergence. We use the software MUSCOD-II, again based on direct multiple shooting, [25]. Note that the F8 control problem has several local minima, several are listed on [27]. We use the best known solution as initialization for our study. It results in an optimal integer control

$$v^*(t) = \begin{cases} 0.05236 & \text{if } t \in [0, \tau_1] \cup [\tau_2, \tau_3] \\ -0.05236 & \text{if } t \in [\tau_1, \tau_2] \cup [\tau_3, t_\mathrm{f}] \end{cases}$$

with $\tau_1 = 1.135007, \tau_2 = 1.482512, \tau_3 = 3.088809, t_\mathrm{f} = 3.780858$. Figures 5.1 and 5.2 show the optimal trajectories and the competing Hamiltonians.

It is interesting to note that Outer Convexification can be applied to Problem 5.1.

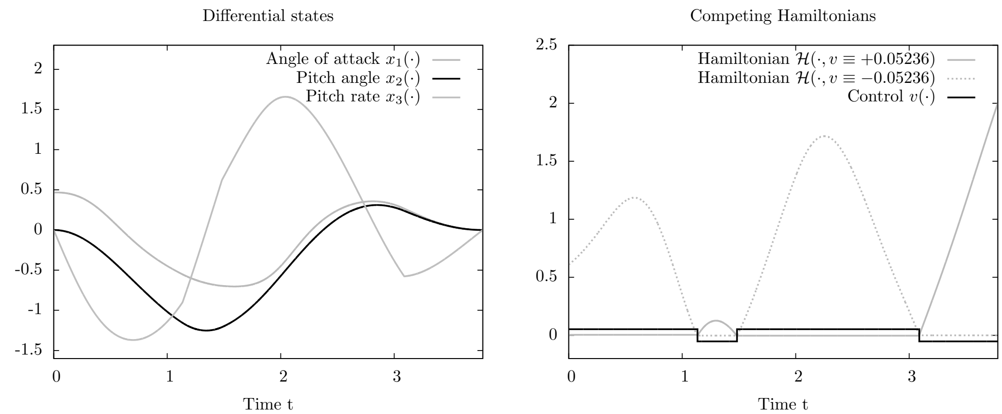

FIG. 5.1. *Left: differential states of the optimal trajectory. Right: the two competing Hamiltonians and the optimal control. Note that the minimizing Hamiltonian* (5.18) *is identical* 0*, in accordance to its end value* (5.17) *and the requirement to be constant, Theorem 3.5 e).*

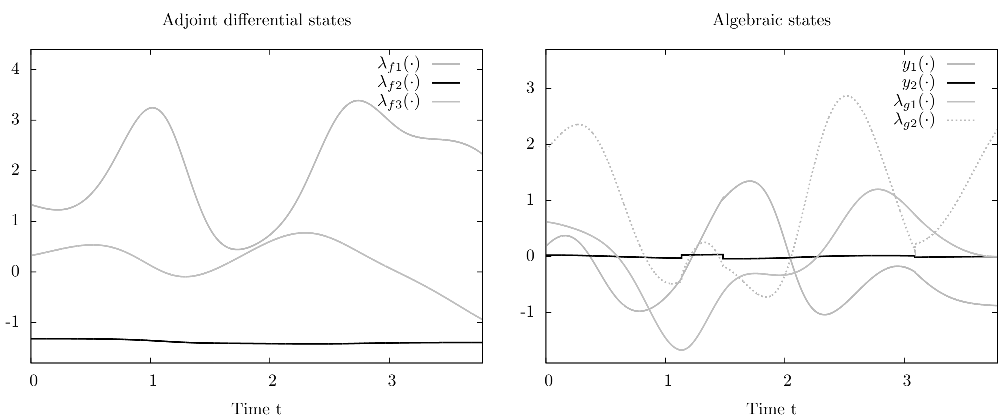

FIG. 5.2. *Left: the adjoint states plotted over time. Right: the algebraic states. The discontinuity in $v^*(\cdot)$ is accounted for by jumps in $y_2(\cdot)$.*

With $\gamma = 0.05236$, the constraints (5.1–5.9) become

$$\dot{x}_1 = -0.877\, x_1 + x_3 - 0.088\, x_1\, x_3 + 0.47\, x_1^2 - 0.019\, x_2^2 - x_1^2\, x_3 \tag{5.19}$$
$$+3.846\, x_1^3 + 0.215\, \gamma - 0.28\, x_1^2\, \gamma + 0.47\, x_1\, \gamma^2 - 0.63\, \gamma^3 \tag{5.20}$$
$$-\left(0.215\, \gamma - 0.28\, x_1^2\, \gamma - 0.63\, \gamma^3\right)\, 2\omega \tag{5.21}$$
$$\dot{x}_2 = x_3 \tag{5.22}$$
$$\dot{x}_3 = -4.208\, x_1 - 0.396\, x_3 - 0.47\, x_1^2 - 3.564\, x_1^3 \tag{5.23}$$
$$+20.967\, \gamma - 6.265\, x_1^2\, \gamma + 46\, x_1\, \gamma^2 - 61.4\, \gamma^3 \tag{5.24}$$
$$-\left(20.967\, \gamma - 6.265\, x_1^2\, \gamma - 61.4\, \gamma^3\right)\, 2\omega \tag{5.25}$$
$$\omega(t) \in \{0, 1\}, \tag{5.26}$$
$$x(0) = (0.4655, 0, 0)^\top, \quad x(t_\mathrm{f}) = (0, 0, 0)^\top \tag{5.27}$$

Any trajectory that minimizes t_f subject to (5.19–5.27) is also an optimal solution to Problem 5.1. According to Corollary 4.3 the objective function value is identical to the one of the relaxed version with $\omega(t) \in [0, 1]$. In this particular case, there doesn't seem to be an integer gap to the nonlinear relaxation (Problem 5.1 with $v(t) \in [-0.05236, 0.05236]$). Examples for integer gaps can be found in [28, 31].

6. Summary. We discussed mixed–integer optimal control problems constrained by differential-algebraic equations of index one. We derived a global minimum principle, making use of a time transformation technique. We also established theoretical results on the integer gap between the objective function value of the original MIOCP and of a relaxed, convexified version which is easier to solve with gradient–based methods. We illustrated the theoretical results by means of a challenging MIOCP benchmark problem. Future work is necessary for DAE systems that do not have index one.

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