Numerical Solution of a Conspicuous Consumption Model with Constant Control Delay

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Abstract

We derive optimal pricing strategies for conspicuous consumption products in periods of recession. To that end, we formulate and investigate a two-stage economic optimal control problem that takes uncertainty of the recession period length and delay effects of the pricing strategy into account.

This non-standard optimal control problem is difficult to solve analytically, and solutions depend on the variable model parameters. Therefore, we use a numerical result-driven approach. We propose a structure-exploiting direct method for optimal control to solve this challenging optimization problem. In particular, we discretize the uncertainties in the model formulation by using scenario trees and target the control delays by introduction of slack control functions.

Numerical results illustrate the validity of our approach and show the impact of uncertainties and delay effects on optimal economic strategies. During the recession, delayed optimal prices are higher than the non-delayed ones. In the normal economic period, however, this effect is reversed and optimal prices with a delayed impact are smaller compared to the non-delayed case.

Key words: Optimal control; Optimization under uncertainties; Control delays; Economic system; Recession

1 Introduction

We are interested in optimal pricing strategies for conspicuous consumption products in periods of recession, such as the credit crunch recession that started in 2007. Besides a reduction in demand, which is quite usual for a recession, in the credit crunch recession capital markets cease to function. Hence firms cannot borrow or issue new shares to finance their operations. They need to self-finance their investments (Economist, 2008):

"...the only option is to try to ride out the recession. But companies can do this only if they have enough liquidity...".

For conspicuous goods demand does not only depend on price, but in addition it depends on the good's reputation, which increases in price. The product's reputation as being expensive allows people to signal their wealth to observers, which in turn increases the reputation of the consumer. Examples of conspicuous goods are lux-

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ury hotels (Times, 2008), expensive cars, or fashionable clothes. The topic of how to price conspicuous goods is treated in Amaldoss & Jain (2005a,b); Kort, Caulkins, Hartl & Feichtinger (2006).

This paper treats the management of conspicuous goods during the credit crunch recession. The conspicuous goods' manager faces the following trade off. To keep future demand at a high level the manager likes to keep the price of its conspicuous good high. However, during the recession demand as such is low and pricing the good high makes demand even lower. This has detrimental effects for the firm's cash flow, which can bring it into bankruptcy problems, because during the recession capital markets do not function so that the firm needs to have a positive cash level in order to prevent bankruptcy. In Caulkins, Feichtinger, Grass, Hartl, Kort & Seidl (2010a, 2011) this problem was extensively analyzed.

The present paper extends Caulkins et al. (2010a, 2011) by establishing a new numerical methodology and by considering a delayed effect of the current price on the firm's reputation. This implies that the good's reputation, which has been built up in the past, is not immediately affected by a price decrease. It takes some time for consumers to get used to the new situation, before a price change really starts to have an effect on the good's reputation.

The very first paper including a delay in an economic model was Kalecki (1935) treating a descriptive business cycle model. Much later, El-Hodiri, Loehman & Whinston (1972) analyzed an optimal growth model with time lags. Starting with the nineties several so-called time-to-build (investment gestation lag) models have been dealt with. Continuous-time deterministic optimal growth models have been enriched by assuming that production occurs with a delay while new capital is installed; see Asea & Zak (1999); Boucekkine, Licandro, Puch & del Rio (2005); Bambi (2008); Bambi, Fabbri & Gozzi (2009); Collard, Licandro & Puch (2008). The methodological background are functional differential equations; for a modified version of Pontryagin's Maximum Principle compare Kolmanovskii & Myshkis (1992). Additionally, in Winkler (2004); Winkler, Brandt-Pollmann, Moslener & Schlöder (2005) some related results are presented. In Collard et al. (2008) economic models characterized by advanced or delayed time arguments in both the states and controls are discussed. The authors present an algorithm combining the method of steps and a specially tailored shooting method.

It turns out that introducing this delayed effect has considerable qualitative implications for pricing the conspicuous good. In particular, the delayed consumer reaction makes that it is optimal for the firm to set a higher price during the recession and a lower one during the normal period.

We formulate and investigate a two-stage economic optimal control problem that takes uncertainty of the recession period length and delay effects of the pricing strategy into account. This non-standard optimal control problem is difficult to solve analytically, and solutions depend on the variable model parameters. Therefore we use a numerical result-driven approach. We propose a structure-exploiting direct method for optimal control to solve this challenging optimization problem. In particular, we discretize the uncertainties in the model formulation by using scenario trees and target the control delays by introduction of slack control functions.

The paper is organized as follows: In Section 2 we take a closer look upon the model. We specify the underlying dynamics for each of the economic stages and deduce the objective function. In Section 3 we first collect the algorithmic approaches used to solve a standard multi-stage optimal control problem numerically. Then we reformulate the model using a scenario tree approach and rearrange the emerging scheme to improve performance and simplify the incorporation of the delay via slack control functions. Section 4 treats analytical and numerical results and their economic interpretations in detail.

2 Model Formulation

We consider an economic setting with a recession period followed by a normal economic period. In the following, the value τ will denote the endpoint of the crisis, compare Figure 1.

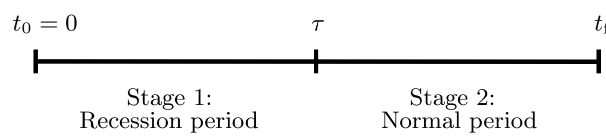


Fig. 1. Stages $[t_0, \tau]$ and $[\tau, t_{\mathrm{f}}]$ of the recession model.

The dynamics of our model includes two states. The brand image A of the firm evolves in both periods according to the differential equation

$$\dot{A}(t) = \kappa(\gamma p(t-\sigma) - A(t)) \tag{1}$$

with a possible constant control delay $\sigma \geq 0$ in the dynamics of the reputation $A(\cdot)$, retarding the connection between changing the price $p(\cdot)$ and its consequence on the development of $A(\cdot)$. Equation (1) covers that, as usual with conspicuous goods, the reputation of the brand goes up with the price, which works positively on demand. Compared to the literature, the delay is a new feature, which captures the fact that consumers first have to get used to a new situation before they adjust their purchase decisions. In particular, if a good is known to be exclusive, a sudden price reduction at first instance does not change this perception. However, after a while consumers "forget" the old situation, implying that they start recognizing that the good is less exclusive, and reputation starts to decrease. Note that if the recession ends at time τ, we still have the direct influence of the price set during the final time interval of length σ of the recession. For a fixed price $\bar{p}$ equation (1) yields a steady state of $\bar{A} = \gamma\, \bar{p}$. The available cash $B(\cdot)$ depends on the gains $p(\cdot)\; D(\cdot)$, fixed costs C, and the short-time interest δ, leading to

$$\dot{B}(t) = p(t)D(A(t), p(t)) - C + \delta B(t).$$

Therein the demand D is driven by the brand image and the pricing strategy $p(\cdot)$, which is the control of our problem. It is essentially influenced by the economic stage, i.e., in the normal period (N) we have

$$D_{\mathrm{N}}(A(t), p(t)) = m - \frac{p(t)}{A(t)^{\beta}}, \tag{2a}$$

whereas in the recession (R) demand is reduced to

$$D_{\mathrm{R}}(A(t), p(t)) = D_{\mathrm{N}}(A(t), p(t)) - \alpha. \tag{2b}$$

The positive constant α measures the strength of the crisis, the parameter $0 < \beta < 1$ is given, and m corresponds to the potential market size.

The objective of the company is to maximize the expected value of profit over the finite or infinite time horizon $[0, t_{\mathrm{f}}]$ of interest. The profit is composed of two parts: the gains of the normal economic period $(\tau, t_{\mathrm{f}}]$ and an impulse dividend of the cash reserve at the end of the recession phase, $B(\tau)$. This dividend is included as the capital market is assumed to become functional again in the normal economic period and firms can freely borrow and lend cash there. Thus, the firm does not need a positive $B(\cdot)$ on $(\tau, t_{\mathrm{f}}]$. For a fixed τ and a given discount rate r, the objective function is calculated as

$$\begin{aligned}\Phi(\tau) := {} & \mathrm{e}^{-r\tau} B(\tau) \\ & + \int_{\tau}^{t_{\mathrm{f}}} \mathrm{e}^{-rt}\left(p(t)D_{\mathrm{N}}(A(t), p(t)) - C\right)\,\mathrm{d}t,\end{aligned} \tag{3}$$

being the sum of these two components, resulting in the optimal control problem

$$\begin{aligned}
\max_{p(\cdot)}\;\; & \Phi(\tau) \\
\text{s.t.}\;\; & \dot{A}(t) = \kappa(\gamma p(t-\sigma) - A(t)), && t \in [0, t_{\mathrm{f}}], \\
& p(t) = \eta(t), && t \in [-\sigma, 0], \\
& \dot{B}(t) = p(t)D_{\mathrm{R}}(A(t), p(t)) \\
& \qquad - C + \delta B(t), && t \in [0, \tau], \\
& A(0) = A_0, \quad B(0) = B_0, \\
& 0 \le D_{\mathrm{R/N}}(A(t), p(t)), && t \in [0, t_{\mathrm{f}}], \\
& p(t) \ge 0, && t \in [0, t_{\mathrm{f}}], \\
& B(t) \ge 0, && t \in [0, \tau]
\end{aligned} \tag{4}$$

with $D_{\mathrm{R/N}}(A(t), p(t))$ given as in (2) and $B(t)$ negligible in the normal period $(\tau, t_{\mathrm{f}}]$. However, typically the recession length τ is not known beforehand to decision makers. An individual firm also has no influence on when the recession ends. Therefore, we assume that the length of the recession period τ is an exponentially distributed random variable. The goal is to maximize the expectation value of the net present value (NPV) at time τ, i.e., the objective function Φ weighted by the exponential probability density function with rate parameter λ,

$$\max_{p(\cdot)}\; \mathbb{E}\left[\mathrm{NPV}(\tau)\right] := \max_{p(\cdot)}\; \int_0^{t_{\mathrm{f}}} \lambda\, \mathrm{e}^{-\lambda\tau}\; \Phi(\tau)\; \mathrm{d}\tau \tag{5}$$

subject to the constraints given in (4) for all $0 \le \tau \le t_{\mathrm{f}}$.

This problem is a non-standard optimal control problem in the sense that uncertainty and control delays are present, making analytical investigations difficult. [1] Therefore, we propose a different approach in the next section.

3 Numerical treatment

We propose to use reformulations to transfer the optimal control problem (5) into a more standard form that can be efficiently solved. In Section 3.1 we present such a standard multi-stage formulation and give references to Bock's direct multiple shooting method. In Section 3.2 we present a discretization of the uncertainty, and in Section 3.3 a reformulation of the time delays. In both cases alternatives are discussed.

[1] In Caulkins, Hartl & Kort (2010b) it is shown that an important class of models with delays can be transformed into equivalent problems without delays. However, the present model does not fit in this family. This is because the control p appears with a delay in one state equation and without in the other one. Hence, it is not possible to eliminate the delay using a time transformation.

3.1 The Direct Multiple Shooting Approach

Efficient numerical methods have been developed to solve multi-stage, nonlinear optimal control problems of the following form

$$\max_{x_i(\cdot),u_i(\cdot),q,t_i} \sum_{i=0}^{M-1} \left\{ \int_{t_i}^{t_{i+1}} L_i(x_i(t), u_i(t), q)\, \mathrm{d}t + E_i(x(t_{i+1}), q) \right\} \tag{6a}$$

$$\text{s.t.} \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t), q), \tag{6b}$$

$$x_{i+1}(t_{i+1}) = f_{\mathrm{tr},i}(x_i(t_{i+1}), q), \tag{6c}$$

$$0 \le c_i(x_i(t), u_i(t), q) \tag{6d}$$

$$0 = r_{\mathrm{eq}}(x_0(t_0), x_1(t_1), \ldots, q), \tag{6e}$$

$$0 \le r_{\mathrm{ineq}}(x_0(t_0), x_1(t_1), \ldots, q), \tag{6f}$$

with $t \in [t_i, t_{i+1}]$ and $i = 0, \ldots, M-1$. The optimization problem (6) couples M model stages via explicit transitions (6c) and interior point constraints (6e-6f). The differential states $x_i : [t_0, t_M] \mapsto \mathbb{R}^{n_{x_i}}$ and the control functions $u_i : [t_0, t_M] \mapsto \mathbb{R}^{n_{u_i}}$ and control values $q \in \mathbb{R}^{n_q}$ need to be feasible for the path- and control constraints (6d) and the ordinary differential equations (ODEs) (6b).

An overview over different methods can be found, e.g., in Binder, Blank, Bock, Bulirsch, Dahmen, Diehl, Kronseder, Marquardt, Schlöder & Stryk (2001). We propose to use Bock's direct multiple shooting method to solve problems of type (6). It transforms the optimal control problem into a Nonlinear Program (NLP) by discretizing the space of admissible control functions $u(\cdot)$ and the path constraints (6d). The solutions of the ODEs (6b) are obtained by a decoupled integration on a multiple shooting grid, starting from artificial intermediate variables. Continuity of the differential states is assured by means of an inclusion of matching conditions into the NLP.

For details on this method we refer to Bock & Plitt (1984); Leineweber (1999); Leineweber, Bauer, Bock & Schlöder (2003). At this place we would only like to remind the reader of one of the advantages of the direct multiple shooting method. As control functions, constraints and multiple shooting variables are discretized on a common time grid, the Hessian of the Lagrangian is block structured for linearly coupled point constraints $r_{\cdot}(\cdot)$. For $i \neq j$ we have

$$\frac{\nabla^2 \mathcal{L}(w_1, \ldots, w_N)}{\partial w_i\, \partial w_j} = 0 \tag{7}$$

for variable vectors w_i that subsume all variables of the i-th multiple shooting interval. This allows applying Broyden–Fletcher–Goldfarb–Shanno (BFGS) updates to every single one of the N multiple shooting blocks (Bock & Plitt, 1984). These high-rank updates typically lead to a fast accumulation of higher order information and thus to fast convergence (Nocedal & Wright, 2006). This feature will become important in the context of the following reformulations of problem (5).

3.2 Discretizing the probability density function

To solve problem (5) at least approximatively, we need to reformulate it. We discretize the exponential distribution of the random variable τ by defining a time grid

$$0 = \tau_0 < \tau_1 < \ldots < \tau_n < t_{\mathrm{f}}.$$

In the following, switches from recession period to normal stage will only be possible at these times τ_i with $i = 1 \ldots n$. The recession ends at τ_i with probability $\mathbb{P}_i$. We use an equidistant discretization, resulting in a geometric distribution

$$\mathbb{P}_i = \int_{\tau_{i-1}}^{\tau_i} \lambda\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t = \mathrm{e}^{-\lambda \tau_{i-1}} - \mathrm{e}^{-\lambda \tau_i}, \tag{8a}$$

for $i = 1, \ldots, n-1$, and

$$\mathbb{P}_n = 1 - \sum_{j=1}^{n-1} \mathbb{P}_j. \tag{8b}$$

The discretized distribution can be used to reformulate the maximization of the expected value as a multi-stage optimal control problem of type (6), by using a scenario tree. However, this formulation is not unique. One possibility is to use a staircase-like approach, increasing the number of variables as the number of possible recession ends τ_i increases. This approach is illustrated schematically in Figure 2 and results in $M = n+1$ model stages, where n is the number of discretizations of the probability density function. The dimensions $n_{x_i} = 2 + i$ of differential states and $n_{u_i} = 1 + i$ of control functions, $i = 0, \ldots, M-1$, are different on the model stages. The transition functions (6c) are defined by

$$A_{i,j}(\tau_i) = A_{i-1,j}(\tau_i), \quad 1 \le j \le i, \tag{9a}$$

$$A_{i,i+1}(\tau_i) = A_{i-1,1}(\tau_i), \tag{9b}$$

$$B_{i,1}(\tau_i) = B_{i-1,1}(\tau_i), \tag{9c}$$

for all model stages $i = 1 \ldots n-1$, and

$$A_{n,n+1}(\tau_n) = A_{n-1,1}(\tau_n). \tag{9d}$$

At each τ_i one has to distinguish between transitions (9a), (9c) of the brand image A and the cash B for the ongoing recession and the initialization (9b), (9d) of the additional differential states $A_{i,i+1}$ for the normal period beginning at τ_i, compare Figure 2.

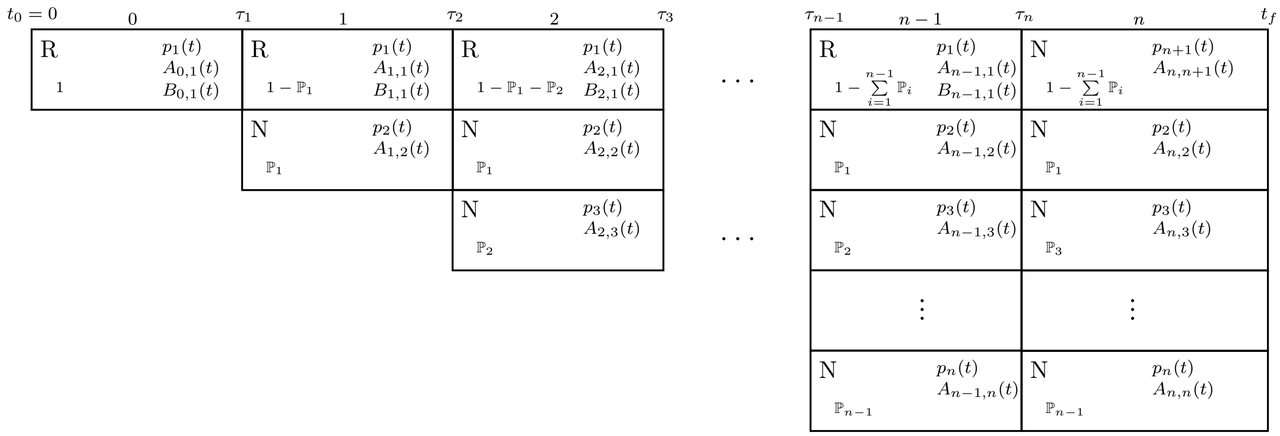


Fig. 2. Controls and variables in the multi-stage formulation of problem (4) with associated probabilities and in a (R)ecession or a (N)ormal period.

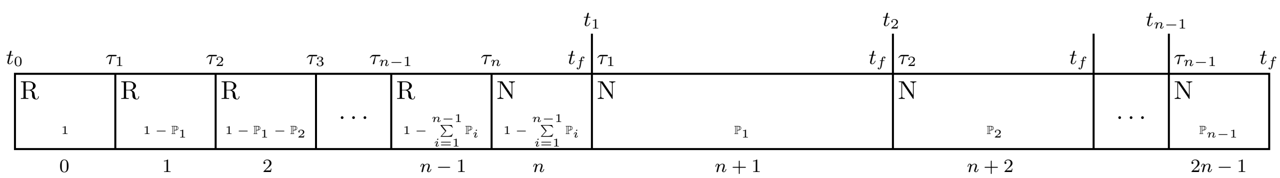


Fig. 3. Rearranged scheme for the discretization of the random end time τ of the recession. Again, the symbols denote the (R)ecession and (N)ormal stage, as well as the appropriate probabilities.

The second possibility is to use linearly coupled point constraints of type (6e) instead of transitions to initialize the new variables. All possible scenarios at τ_i are concatenated, resulting in $M = 2n$ model stages. This "flat" arrangement of stages is shown in Figure 3.

In contrast to the first formulation, the model stage dimensions $n_{x_i} = 2$ for $i = 0, \dots, n-1$ and $n_{x_i} = 1$ for $i = n, \dots, M-1$ of differential states and $n_{u_i} = 1$ for $i = 0, \dots, M-1$ of controls are (almost) constant. The coupled point constraints (6e) are given by

$$A_{i,1}(t_{i-n}) = A_{i-n-1,1}(\tau_{i-n}), \quad n+1 \le i \le 2n-1. \tag{10a}$$

The first n stages are recession periods with continuous transitions of all states. They differ in the objective function. The transition from the last recession stage n to the subsequent normal period that starts at $t = \tau_n$ is continuous, too. However, the model stage lengths of this approach vary. While all n recession stages have the constant duration $h = \tau_i - \tau_{i-1}$, the n normal period stages have a length of $t_\mathrm{f} - \tau_i$, $i = 1, \dots, n$.

Then we obtain for the staircase-like approach to discretize the probability density function, $k = 1$, the objective function

$$\begin{aligned}\Phi_i^1(&\tau_i, A_{i,\cdot}(t), B_{i-1,1}(\tau_i), p(t), \bar{\mathbb{P}}_i)\\ &= \mathbb{P}_i \,\mathrm{e}^{-r\tau_i} B_{i-1,1}(\tau_i) \qquad (11a)\\ &\quad + \sum_{j=1}^{i} \mathbb{P}_j \int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{-rt}\,(p(t) D_\mathrm{N}(A_{i,j+1}(t), p(t)) - C)\,\mathrm{d}t\end{aligned}$$

for $i = 1, \dots, n$, the transition (tr) functions

$$f^1_{\mathrm{tr}A,i}(A_{i-1,j}(\tau_i)) = \begin{cases} A_{i-1,j}(\tau_i); & 1 \le i \le n-1,\ 1 \le j \le i, \\ A_{i-1,1}(\tau_i); & 1 \le i \le n,\ j = i+1, \end{cases} \tag{11b}$$

$$f^1_{\mathrm{tr}B,i}(B_{i-1,1}(\tau_i)) = B_{i-1,1}(\tau_i),\ 1 \le i \le n-1, \tag{11c}$$

and the coupled point constraints functions

$$r^1_{\mathrm{eq},i} \equiv 0, \tag{11d}$$

where $\bar{\mathbb{P}}_i = (\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_i)$.

The concatenated approach, $k = 2$, is defined by the respective functions

$$\begin{aligned}\Phi_i^2(&\tau_i, A_{n+i,1}(t), B_{i-1,1}(\tau_i), p(t), \mathbb{P}_i)\\ &= \mathbb{P}_i \,\mathrm{e}^{-r\tau_i} B_{i-1,1}(\tau_i) \qquad (12a)\\ &\quad + \mathbb{P}_i \int_{\tau_i}^{t_\mathrm{f}} \mathrm{e}^{-rt}\,(p(t) D_\mathrm{N}(A_{n+i,1}(t), p(t)) - C)\ \mathrm{d}t,\end{aligned}$$

for $i = 1, \ldots, n$,

$$f^2_{\mathrm{tr}A,i}(A_{i-1,1}(\tau_i)) = A_{i-1,1}(\tau_i),\ 1 \le i \le n, \tag{12b}$$
$$f^2_{\mathrm{tr}B,i}(B_{i-1,1}(\tau_i)) = B_{i-1,1}(\tau_i),\ 1 \le i \le n-1, \tag{12c}$$

$$\begin{aligned} r^2_{\mathrm{eq},i}&(A_{i,1}(t_{i-n}), A_{i-n-1,1}(\tau_{i-n})) \\ &= A_{i,1}(t_{i-n}) - A_{i-n-1,1}(\tau_{i-n}),\ n+1 \le i \le M-1. \end{aligned} \tag{12d}$$

3.3 Reformulation of the time delays

In Brandt-Pollmann, Winkler, Sager, Moslener & Schlöder (2008) two possibilities are given to reformulate an optimal control problem with delayed equation of motion as in (4) into an instantaneous problem.

The first approach splits the time horizon t_f into m parts of length σ and formulates the system dynamics separately on each of the resulting intervals. By interpreting them as independent and introducing new state and control variables we can formulate a system of m differential equations on the time horizon $[0, \sigma]$. This can be used to reformulate the original optimal control problem. Furthermore, one has to introduce coupled boundary conditions to ensure the continuity of the state variable. The approach may give additional insight from an analytical point of view, compare Brandt-Pollmann et al. (2008). However, it requires the determination of $m-1$ control paths in the interval $[0, \sigma]$. For small values of the delay σ this results in a large number of state and control functions.

Therefore, we prefer a different reformulation. We introduce a second control function $u_2(t) = p(t)$ that denotes the unretarded control at time t, whereas $u_1(t) = p(t-\sigma)$ characterizes the delayed one. They are coupled via equalities $u_1(t) = u_2(t-\sigma)$ for $t \ge \sigma$ and $u_1(t) = \eta(t-\sigma)$ for $0 \le t \le \sigma$.

Taking either staircase (11) or flat (12) discretization of uncertainty presented in the previous Section, $k = 1, 2$, we obtain

$$\max_{u_1(\cdot),u_2(\cdot)} \sum_{i=1}^{n} \Phi^k_i(\tau_i, A_{\chi^k(i),\cdot}(t), B_{i-1,1}(\tau_i), u_2(t), \bar{\mathbb{P}}_i) \tag{13a}$$

$$\begin{aligned}
\text{s.t. } \dot{A}_{i,j}(t) &= \kappa(\gamma u_1(t) - A_{i,j}(t)), && t \in [0, t_\mathrm{f}], && \text{(13b)}\\
&&& 0 \le i \le M-1,\ j \in J^k, \\
\dot{B}_{i,1}(t) &= u_2(t) D_\mathrm{R}(A_{i,1}(t), u_2(t)) &&&& \text{(13c)}\\
&\quad - C + \delta B_{i,1}(t), && t \in [0, \tau_i], \\
&&& 0 \le i \le n-1, \\
u_1(t) &= \eta(t-\sigma), && t \in [0, \sigma], && \text{(13d)}\\
u_1(t) &= u_2(t-\sigma), && t \in [\sigma, t_\mathrm{f}], && \text{(13e)}\\
A_{0,1}(0) &= A_0, \quad B_{0,1}(0) = B_0, \\
0 &\le D_{\mathrm{R,N}}(A_{i,j}(t), u_2(t)), && t \in [0, t_\mathrm{f}], && \text{(13f)}\\
u_1(t) &\ge 0, \quad u_2(t) \ge 0, && t \in [0, t_\mathrm{f}], && \text{(13g)}\\
B_{i,1}(t) &\ge 0, && t \in [0, \tau_i], && \text{(13h)}\\
&&& 1 \le i \le n-1, \\
A_{i,j}(\tau_i) &= f^k_{\mathrm{tr}A,i}(A_{i-1,j}(\tau_i)), &&&& \text{(13i)}\\
&&& 1 \le i \le n,\ j \in J^k, \\
B_{i,1}(\tau_i) &= f^k_{\mathrm{tr}B,i}(B_{i-1,1}(\tau_i)), &&&& \text{(13j)}\\
&&& 1 \le i \le n-1, \\
0 &= r^k_{\mathrm{eq},i}(A_{i,1}(t_{i-n}), A_{i-n-1,1}(\tau_{i-n})), &&&& \text{(13k)}\\
&&& n+1 \le i \le M-1,
\end{aligned}$$

where $\chi^1(i) := i$, $\chi^2(i) := n+i$, $J^1 := \{j \mid 1 \le j \le i+1\}$, $J^2 := \{j \mid j = 1\}$.

This problem still contains a delayed term, but it is not apparent in the system dynamics anymore. It has moved to a constraint (13e) on the controls. This can be efficiently dealt with the multiple shooting method we introduced in Section 3.1 for the special case of a constant delay.

4 Results

As suggested in Caulkins et al. (2010a, 2011), we use the following set of parameters in our numerical treatment:

$$\begin{aligned}
\kappa &= 2.0, & \gamma &= 5.0, & C &= 7.5, & \delta &= 0.05, \\
m &= 3.0, & \beta &= 0.5, & r &= 0.1, & \lambda &= 0.5, \\
\alpha_1 &= 0.7, & \alpha_2 &= 0.836, & \alpha_3 &= 1.25.
\end{aligned} \tag{14a}$$

The choice for parameters r, δ, and λ is based on the assumption that we measure time in years and that the expected duration of the recession is two years. We set β assuming that an increase in reputation will influence less and less customers. The more fashionable the product is, the more specialized is its market niche. See Caulkins et al. (2011) for a motivation of the remaining parameters.

A key result of Caulkins et al. (2011) was that the authors were able to distinguish three different types of recessions corresponding to the severity of the demand reduction and the resulting optimal strategy. Following their results, the values of the parameter α indicate a mild ($\alpha_1 = 0.7$), intermediate ($\alpha_2 = 0.836$), and severe ($\alpha_3 = 1.25$) economic crisis.

Due to the discretization of τ we need to further specify the last possible endpoint of the recession,

$$\tau_n = 20. \tag{14b}$$

This implies that in this context the probability that the recession persists longer than that is low, i.e., $\mathbb{P}[\tau > 20] = 4.54 \cdot 10^{-5}$. For the control delay we choose

$$\sigma = 0.25. \tag{14c}$$

To accomplish this, two equidistant discretization step lengths are applied, first with $n_1 = 20$, i.e., $h = \tau_i - \tau_{i-1} = 1.0$, and $n_2 = 40$, i.e., $h = 0.5$. Each of them is combined with four shooting nodes per one time unit, i.e., per one year. Then condition (13e) can be implemented via interior point constraints applied on the shooting nodes.

For convenience, the overall final time t_f is chosen to be

$$t_\mathrm{f} = 21 \text{ (years)}, \tag{14d}$$

so that we definitely have a small normal period of one year in all possible stages.

Finally, in the subsequent sections we provide some computational results. They are obtained with the following combinations of number of discretization points n, recession parameter α, initial values (A_0, B_0), and initial price paths η for the delayed model, cf. Table 1.

In Section 4.1 we analyze the computational performance of the various reformulations presented in the previous section. In Section 4.2 we derive some analytical insight into the problem structure. More economic insight can be gained from the computational results in Section 4.3.

4.1 Computational performance

As discussed in Sections 3.2 and 3.3 different mathematically equivalent reformulations of the optimal control problem (4) exist. However, they are by no means equivalent from a computational point of view.

Table 2 compares the computational performance of the two different approaches to discretize the uncertainty. With the staircase formulation (11) (Figure 2) the overall time horizon is quite small. However, the number of state variables is increased compared to the concatenated arrangement, leading to more steps of the error-controled, adaptive integrator. More significant, however, is the impact of more blocks in the Hessian of the

Scenario	n	α	A_0	B_0	η
1	20	0.7	10.0	5.0	-
2	20	0.836	20.0	5.0	-
3	20	1.25	100.0	100.0	-
4	40	0.7	10.0	5.0	7.406785
5	40	0.7	0.1	5.0	4.296460
6	40	0.7	10.0	2.0	7.088001
7	40	0.7	$\bar{A}_\mathrm{d}^\mathrm{N}$	5.0	$\bar{p}_\mathrm{d}^\mathrm{N}$
8	40	0.7	$\bar{A}_\mathrm{d}^\mathrm{N}$	1.0	$\bar{p}_\mathrm{d}^\mathrm{N}$
9	40	0.7	$\bar{A}_\mathrm{d}^\mathrm{N}$	0.1	$\bar{p}_\mathrm{d}^\mathrm{N}$
10	40	0.836	0.1	10.0	3.917962
11	40	0.836	0.1	10.0	3.5
12	40	0.836	0.1	10.0	3.0
13	40	0.836	0.1	10.0	2.5
14	40	0.836	20.0	5.0	8.153575
15	40	0.836	0.1	8.0	3.917948
16	40	0.836	25.0	3.5	8.671824
17	40	0.836	$\bar{A}_\mathrm{d}^\mathrm{N}$	1.0	$\bar{p}_\mathrm{d}^\mathrm{N}$
18	40	0.836	0.1	7.05	-
19	40	0.836	63.0	0.05	-
20	40	0.836	0.1	9.8	3.5
21	40	0.836	73.5	0.1	12.517549
22	40	1.25	100.0	100.0	10.751307
23	40	1.25	0.1	100.0	2.924618
24	40	1.25	40.0	80.0	7.855208
25	40	1.25	80.0	50.0	9.922934
26	40	1.25	0.1	60	2.924617
27	40	1.25	$\bar{A}_\mathrm{d}^\mathrm{N}$	50.0	$\bar{p}_\mathrm{d}^\mathrm{N}$
28	40	1.25	$\bar{A}_\mathrm{d}^\mathrm{N}$	70.0	-
29	40	1.25	0.1	76.0	-
30	40	1.25	$\bar{A}_\mathrm{d}^\mathrm{N}$	71.5	$\bar{p}_\mathrm{d}^\mathrm{N}$
31	40	1.25	0.1	79.5	2.924580

Table 1. Different scenarios used for computational performance tests and visualizations. Note that some of these scenarios are used in both a delayed ($\sigma = 0.25$) and undelayed model ($\sigma = 0$), others in only one of them. In undelayed settings η is obsolete and denoted by "-".

Lagrangian. They are used for high-rank updates, compare Section 3.1. This leads to a drastic increase in local convergence and hence to a decrease of the number of sequential quadratic programming (SQP) iterations (Leineweber et al., 2003) and overall computation time, as can be seen in Table 2 for the case $\sigma = 0$. These results

	Scheme (11)		Scheme (12)	
Scenario	# of SQP	t (sec.)	# of SQP	t (sec.)
1	846	5259	51	1341
2	829	1312	35	835
3	858	1411	102	2969
4	1254	67131	102	21443
14	1716	93773	48	9615
22	915	47285	102	24163

Table 2. Comparison of the different schemes for discretizing τ, see (11), (12), and Figures 2, 3, respectively. The results correspond to the undelayed case, i.e., $\sigma = 0$. The faster convergence of (12) (recognizable in SQP iterations and runtime) is due to the high-rank updates mentioned in Section 3.1. The scenarios are listed in Table 1.

	Undelayed model		Delayed model	
	$n = 20$	$n = 40$	$n = 20$	$n = 40$
discr. points	940	1840	940	1840
variables	3797	7437	4738	9278
eq. constraints	2855	5595	3797	7437
ineq. constraints	7594	14874	9476	18556

Table 3. Comparison of the size of the resulting NLP for the delayed and the undelayed model.

carry over to the case with $\sigma > 0$, therefore we will concentrate on the formulation (12) visualized in Figure 3.

As already observed in Brandt-Pollmann et al. (2008), the first approach suggested in Section 3.3 to handle time lags σ is computationally inferior to the second one, although it might be interesting from an analytical point of view. E.g., for scenarios 4–12 the number of 1800 additional state and 1799 control functions needs to be included. Therefore, we will use the second formulation in the following for our calculations. Table 3 gives an overview over the moderate increase in the dimension of the resulting nonlinear program.

Table 4 gives an indication of the computational expense for including delays. The main part of the computation is needed for the condensing algorithm, see Bock & Plitt (1984); Leineweber (1999), which is almost identical for both cases, as the state dimension is independent of σ. The main extra cost is solving the quadratic programs, as the runtime depends crucially on the number of control variables. Therefore, asymptotically for $\sigma > 0$ getting smaller and smaller, the quadratic programming (QP) runtime will become more and more dominant.

	Undelayed model		Delayed Model	
Scenario	# of SQP	t (sec.)	# of SQP	t (sec.)
6	71	14103	60	20238
7	102	24515	98	28422
16	70	12896	102	28787
17	69	14796	82	24466
24	81	18114	81	22166
27	101	24456	101	29404

Table 4. Number of iterations and CPU time for undelayed and delayed scenarios. The computational effort is moderately higher, when delays are taken into account.

4.2 Analytical results

We deduce analytical results that help us to obtain a better insight into the qualitative changes related to the introduction of the time lag σ. We investigate the steady state in the normal period of our model (4) and compare it with the result of the undelayed case, i.e., $\sigma = 0$.

The integral term of $\Phi(\cdot)$ in (3) corresponds to the normal economic period, where the capital markets are working again and we are not using the cash state B anymore. Let $\bar{A}^{\mathrm{N}}_{\mathrm{d/nd}}$ and $\bar{p}^{\mathrm{N}}_{\mathrm{d/nd}}$ denote the normal period's steady state brand image and price in the (d)elayed and the u(nd)elayed case, respectively.

By using Pontryagin's Maximum Principle (Grass, Caulkins, Feichtinger, Tragler & Behrens, 2008) we calculate

$$\bar{A}^{\mathrm{N}}_{\mathrm{nd}} = \left(\frac{\gamma m(r+\kappa)}{2(r+\kappa)-\beta\kappa}\right)^{\frac{1}{1-\beta}}, \quad \bar{p}^{\mathrm{N}}_{\mathrm{nd}} = \frac{\bar{A}^{\mathrm{N}}_{\mathrm{nd}}}{\gamma}. \tag{15a}$$

In the model's delayed version the maximum principle is far more complex, see El-Hodiri et al. (1972). However, in the normal period the stationary state of the corresponding one-dimensional problem can be derived using the results in Winkler, Brandt-Pollmann, Moslener & Schlöder (2003). We substitute

$$F(t) := F(A(t), p(t)) = p(t)\left(m - \frac{p(t)}{A(t)^{\beta}}\right) - C$$

and obtain the Hamiltonian

$$\mathcal{H} = \mathrm{e}^{-rt}F(t) + \mu(t+\sigma)\cdot\kappa\gamma p(t) - \mu(t)\cdot\kappa A(t)$$

with the co-state variable $\mu(t)$. This induces the system

$$\dot{A}(t) = \kappa(\gamma p(t-\sigma) - A(t))$$
$$\dot{p}(t) = \frac{1}{F_{pp}(t)}\big((r+\kappa)F_p(t) + \kappa\gamma\,\mathrm{e}^{-r\sigma} F_A(t+\sigma) - F_{pA}(t)\dot{A}(t)\big)$$

that directly gives us the stationary price $\bar{p}_{\mathrm{d}}^{\mathrm{N}}$. Further on, it yields

$$\frac{(r+\kappa)\,\mathrm{e}^{r\sigma}}{\kappa\gamma} = -\frac{F_A(t+\sigma)}{F_p(t)}$$

and, therefore, the equality

$$(r+\kappa)\,\mathrm{e}^{r\sigma} = -\frac{\beta\kappa(\bar{A}_{\mathrm{d}}^{\mathrm{N}})^{1-\beta}}{\gamma m - 2(\bar{A}_{\mathrm{d}}^{\mathrm{N}})^{1-\beta}}$$

that determines the stationary state of the brand image

$$\bar{A}_{\mathrm{d}}^{\mathrm{N}} = \left(\frac{\gamma m(r+\kappa)\,\mathrm{e}^{r\sigma}}{2(r+\kappa)\,\mathrm{e}^{r\sigma} - \beta\kappa}\right)^{\frac{1}{1-\beta}}, \quad \bar{p}_{\mathrm{d}}^{\mathrm{N}} = \frac{\bar{A}_{\mathrm{d}}^{\mathrm{N}}}{\gamma}. \tag{15b}$$

The latter result obviously includes the special case (15a). Our parameters (14) determine the values

$$\bar{A}_{\mathrm{nd}}^{\mathrm{N}} = 96.899414, \qquad \bar{p}_{\mathrm{nd}}^{\mathrm{N}} = 19.379883, \tag{16a}$$
$$\bar{A}_{\mathrm{d}}^{\mathrm{N}} = 95.421259, \qquad \bar{p}_{\mathrm{d}}^{\mathrm{N}} = 19.084252. \tag{16b}$$

Those coincide with the numerical results we obtained. One can see the impact of the delay very clearly. The benefit of keeping the price up is obtained later in the delayed world, while the benefit of reducing it (with instantaneous profit) is still obtained immediately.

In the recession period the verification and calculation of steady states cannot be done this straightforwardly. Further on, the so-called weak Skiba curves [2] play an important role. While the authors of Caulkins et al. (2010a) were able to derive several results of the non-delayed case analytically, for the delayed model this is impeded much more.

4.3 Computational results

In our approach to discretize problem (4) we assume a finite and discrete grid of possible switching times τ_i. We think that this transformation to the finite-time case is well justified, as the influence of the errors caused by the discretization are small. The intervals between τ_i are short and the probability (8b) for switching the stage at the last possible time τ_n is only marginally higher than it would be in the infinite case.

In Caulkins et al. (2010a) possible pricing strategies in recession periods are explained depending on the value of α. Additionally, the impact of these pricing policies on the development of the reputation A and the cash B is depicted. In the delayed world the behavior of the firm is qualitatively similar. In a severe crisis ($\alpha_3 = 1.25$) the brand image and/or cash required to avoid bankruptcy are particularly large. The milder the crisis is the less reputation/cash is needed. In all cases the cash state diverges to infinity if the firm survives with certainty.

The main result of our analysis of problem (4) is the relation

$$p_{\mathrm{d}}(t) > p_{\mathrm{nd}}(t),\ 0 \le t \le \tau, \quad p_{\mathrm{d}}(t) < p_{\mathrm{nd}}(t),\ \tau \le t \le t_{\mathrm{f}}, \tag{17}$$

which can be seen in Figure 4.

The optimal solution of the normal period follows the results of Section 4.2. Due to the delay σ there is a less direct effect of the price p_{d} on the dynamics of the brand image $\dot{A}$. This reduces the incentive to set a high price, as a lower price raises revenues, which consequently raises the value of the objective function immediately.

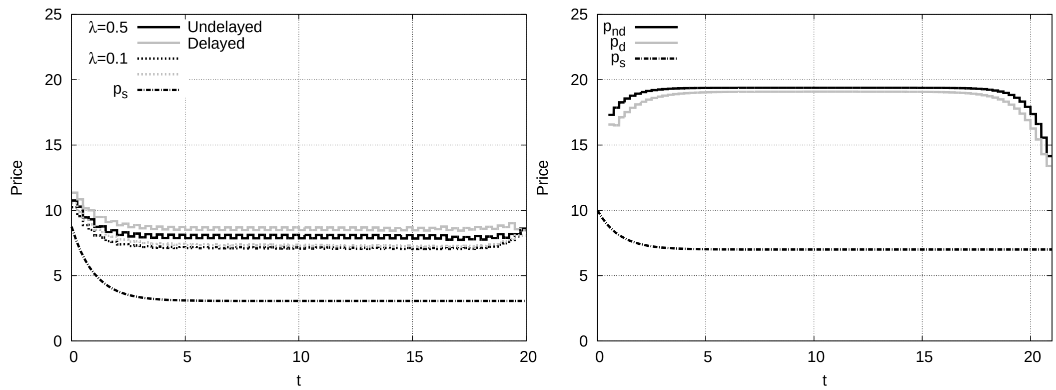
In the recession period, however, the opposite relation holds. A direct consequence of this is visible in Figures 5 and 6: The vertical line indicating the divergence of the cash state B in an infinite horizon setting is shifted to a value $\bar{A}_{\mathrm{d}}^{\mathrm{R}}$ of reputation that is higher than the respective value $\bar{A}_{\mathrm{nd}}^{\mathrm{R}}$ in the non-delayed case.

While the negative effect of smaller revenues with higher prices (independent of the economic period) is the same for both the delayed and the undelayed case, there are also two positive aspects of increasing the price p_{d}.

The first effect is that the brand image A will increase as well during the recession, implying that the bankruptcy probability reduces. This effect is stronger the less the delay σ is. Hence, this first impact is the strongest in the non-delayed case.

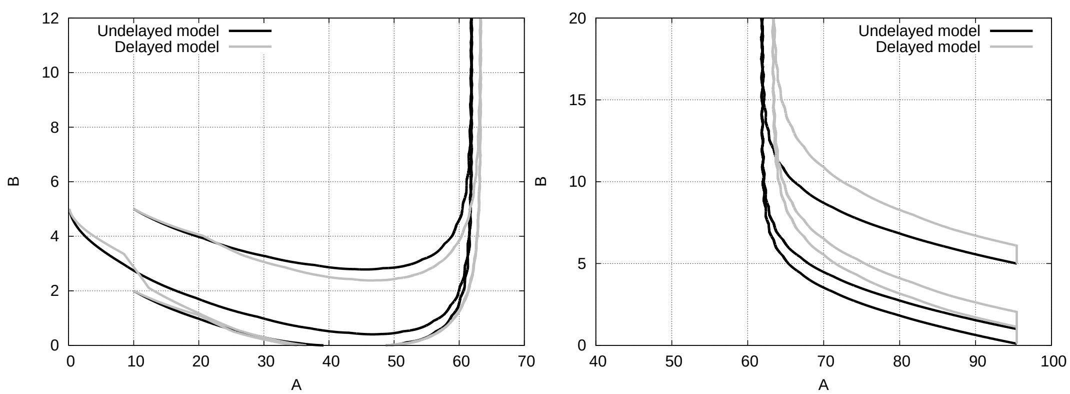
Given that the recession will be terminated somewhere during the next time interval of duration σ, the second effect of increasing p_{d} is that the reputation goes up after the recession, implying that the revenue of the normal period rises. This effect occurs with the probability $\mathbb{P}[\tau \in [t, t+\sigma]]$ that the recession will be over during the next interval of length σ, hence, it is stronger the larger the delay is. But it is completely absent in the undelayed case.

[2] Also known as threshold or weak DNSS curve referring to early contributions of Dechert & Nishimura (1983), Sethi (1977, 1979), and Skiba (1978); see also Grass et al. (2008). Weak Skiba refers to the threshold property of this curve separating different long-term solutions. Which strategy has to be applied is history-dependent and, thus, particularly depends on the initial state values.



(a) Recession in $[0, \tau_n]$ (b) Normal period in $(\tau_1, t_\mathrm{f}]$

Fig. 4. Exemplary price paths of
(a) a recession period lasting until τ_n (using Scenario 22). During the recession $p_\mathrm{d} > p_\mathrm{nd}$ holds, but the difference in between depends on the size of the rate parameter λ.
(b) a normal economic stage for the same scenario setting. By way of better illustration this figure shows price paths of a normal period beginning already at time τ_1. Note that neither λ nor the strength α of the recession have any influence on these paths. For comparison, p_s shows the static optimization price.



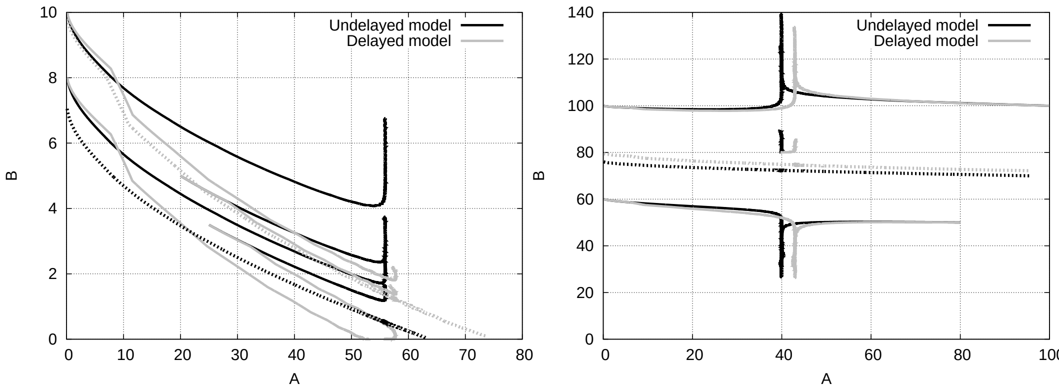
(a) (b)

Fig. 5. Evolution of optimal trajectories over time in a phase diagram with brand image $A(\cdot)$ and capital $B(\cdot)$. They start in (A_0, B_0) according to Table 1 and evolve until $(A(\tau_n), B(\tau_n))$. Optimal solutions of a delayed ($\sigma = 0.25$) and the undelayed ($\sigma = 0$) model are shown for a mild recession ($\alpha_1 = 0.7$), if we assume that for $t \in [-\sigma, 0]$
(a) the recession has been present (Scenarios 4–6 (from top to bottom)),
(b) a steady state normal economic period existed, i.e., $A_0 = \bar{A}_\mathrm{d}^\mathrm{N}$, $\eta = \bar{p}_\mathrm{d}^\mathrm{N}$ (Scenarios 7–9).
Due to the introduction of the delay the recession's steady state of the brand image $\bar{A}_\mathrm{d}^\mathrm{R}$ (and correspondingly $\bar{p}_\mathrm{d}^\mathrm{R}$) is greater than in the undelayed case.

According to the first effect, which is comparable to the impact in the normal period, it will hold that $p_\mathrm{d} < p_\mathrm{nd}$ then. The second effect will imply the opposite relation during the recession stage. Note that this second impact only occurs with $\mathbb{P}[\tau \in [t, t+\sigma]]$, i.e., it depends on the size of σ and the probability density function.

In our case (with $\sigma = 0.25$) the second effect dominates, meaning that the mentioned probability is large enough. For the first effect to dominate we have to decrease this probability by either reducing the time lag or end of recession probability parameter λ. The results of the latter possibility can be seen in Figure 4a.

In a more vivid way we can interpret this second effect by assuming that the crisis ends at time $\hat{\tau}$. In the undelayed case the firm can start building up their reputation immediately after the realization of $\hat{\tau}$ by charging higher prices (supposing that it has survived). The effect on A comes directly. If $\sigma > 0$ the impact of rising prices after $\hat{\tau}$ only starts to have a positive outcome from time $\hat{\tau} + \sigma$ onwards. In the initial phase of the normal period



(a) $\alpha_2 = 0.836$ (b) $\alpha_3 = 1.25$

Fig. 6. Phase diagram as in Figure 5a for an intermediate and severe recession.
(a) Scenarios 10, 14–16, (b) Scenarios 22–26.
In analogy to weak Skiba curves, the dotted lines based on Scenarios (a) 18–21, (b) 28–31 indicate the initial values which separate the state space into the ones (above) that do not lead to bankruptcy and the ones (below) that do. After the introduction of the time lag σ the bankruptcy region becomes larger. This results in an upwards-adjustment of the weak Skiba curve in the delayed case.

$[\hat{\tau}, \hat{\tau} + \sigma]$ the demand is directly influenced by the price set in the last interval of the recession. Hence, increasing prices in $[\hat{\tau} - \sigma, \hat{\tau}]$ leads to a higher reputation σ time units later. I.e., the demand is also higher in the period $[\hat{\tau}, \hat{\tau} + \sigma]$, which generates higher revenues during the first phase of the normal period. As the firm does not know beforehand when the recession will be over, there is always a positive probability that the current time t is located in the period $[\hat{\tau} - \sigma, \hat{\tau}]$. Keeping this in mind, the firm has an additional incentive to keep prices up in recession periods when a delay is apparent, avoiding to damage the reputation too much. Otherwise their product will still perceived to be comparatively cheap for some time period after the recession is over.

Another important result can be observed in Figure 6. As observed in Caulkins et al. (2011), in cases of an intermediate or severe recession there is a weak Skiba curve separating the regions of possible bankruptcy and certain survival. If $\sigma > 0$ this curve is adjusted upwards to some extent. With the incorporation of the delay in our model it is less easy for the firm to survive the crisis because the effect of changing the price p on the brand image is less direct. This explains why the bankruptcy region becomes larger.

At the end of this Section we want to remark that the condition (13d) causes two main scenarios we have to distinguish in the delayed model. The economic stage that is apparent in the time prior to the planning period $[0, t_f]$ can either be a normal or a recession stage. We consider two slightly simplified cases.

In the first one we assume a steady state corresponding to the normal economic period in the interval $[-\sigma, 0]$, i.e., we have already one "switching" occurrence at the beginning of the horizon. We initialize the retarded control with $\eta = \bar{p}_{\mathrm{d}}^{\mathrm{N}}$ and the brand image with $A_0 = \bar{A}_{\mathrm{d}}^{\mathrm{N}}$. Then the system evolves as shown in Figure 5b. The non-smooth behavior of the trajectories there is quite natural. At $t = 0$ the recession begins and the demand is reduced immediately due to the influence of α. Hence, prices will drop and the firm's cash decreases. However, the brand image in the time interval $[0, \sigma]$ develops according to the high steady state price $\bar{p}_{\mathrm{d}}^{\mathrm{N}}$, i.e., it remains at its level. Only thereafter the condition (13e) becomes active and the reputation reacts to the lower prices.

The second case is more complicated. If we suppose a persisting recession stage, it is very hard to find a satisfying initialization η for the retarded price in the interval $[0, \sigma]$. In our calculations we started with the optimal price obtained in the first interval of the non-delayed model. This causes the kink in the initial part of the trajectories in Figures 5 and 6. Experiments of varying the value of η changed the amplitude of this deformation slightly, see Figure 7. In this special scenario the different initializations also had a qualitative influence on the bankruptcy probability of the firm. If the combination of brand image and cash moves below the weak Skiba curve, the firm has to face bankruptcy in the long run. This happens for small initial prices, whereas high ones lead to certain survival.

5 Summary

We showed that a constant control delay in a two-stage model of a firm selling conspicuous consumption goods has a qualitative influence on the optimal pricing strategy the firm should apply in periods of economic uncer-

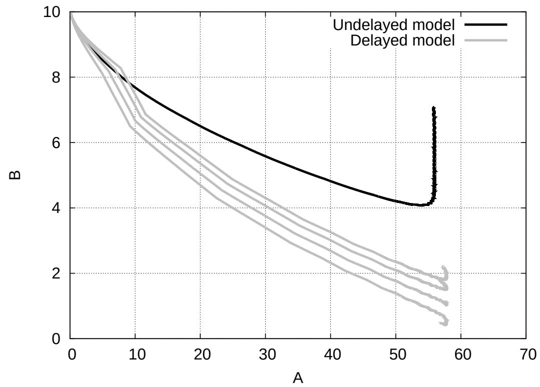


Fig. 7. Phase diagram as in Figure 5a for Scenarios 10–13 (grey lines from top to bottom). It is obvious that the initial control path η has a considerable influence on the firm's future situation.

tainties. In the recession stage of the delayed case the firm should use higher prices than in the corresponding scenario in the undelayed world, whereas in the normal economic stage after the recession is over the pricing policy is optimal if the reversed relation is true. This behavior is strongly depending on the probability that the recession will end during the next σ periods, i.e., on the size of the delay and the rate parameter λ. We also showed that the bankruptcy region is larger if $\sigma > 0$.

Our approach to solve this non-standard optimal control problem by a scenario tree approach deduced from the discretization of the random variable τ as the end point of the crisis and combining this with the introduction of a slack control function to incorporate price delays has proven to be successful. The application of structure-exploiting direct numerical methods is an adequate means to gain insight into solution structures of complex economical systems, also and especially if additional analytical studies are required.

Possible extensions of our model can include state equations with delays in both the control and the state (Collard et al., 2008), the inclusion of quality as additional control, or an reversion of the order of stages, i.e., beginning with a normal period followed by a recession. Another variant can be obtained by a redefinition of the brand image

$$A(t) = \int_{t-\sigma}^{t} p(z)\, \mathrm{d}z,$$

yielding $\dot{A}(t) = p(t) - p(t-\sigma) = p(t) - \delta(t)A(t)$, where the depreciation rate $\delta(t)$ depends on the delayed price (Boucekkine et al., 2005; Collard et al., 2008). Further on, the recession parameter α might be regarded as a random variable as well, possibly even as a random process.

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References

Amaldoss, W., & Jain, S. (2005a). Conspicuous consumption and sophisticated thinking. *Management Science*, *51*, 1449–1466.

Amaldoss, W., & Jain, S. (2005b). Pricing of conspicuous goods: a competitive analysis of social effects. *Journal of Marketing Research*, *42*, 30–42.

Asea, P., & Zak, P. (1999). Time–to–build and cycles. *Journal of Economic Dynamics & Control*, *23*, 1155–1175.

Bambi, M. (2008). Endogenous growth and time to build: the AK case. *Journal of Economic Dynamics and Control*, *32*, 1015–1040.

Bambi, M., Fabbri, G., & Gozzi, F. (2009). Optimal policy and consumption smoothing effects in the time-to-build AK model. MPRA Paper No. 17128, Munich Personal RePEc Archive.

Binder, T., Blank, L., Bock, H., Bulirsch, R., Dahmen, W., Diehl, M., Kronseder, T., Marquardt, W., Schlöder, J., & Stryk, O. (2001). Introduction to model based optimization of chemical processes on moving horizons. In M. Grötschel, S. Krumke, & J. Rambau (Eds.), *Online Optimization of Large Scale Systems: State of the Art* (pp. 295–340). Springer.

Bock, H., & Plitt, K. (1984). A Multiple Shooting algorithm for direct solution of optimal control problems. In *Proceedings of the 9th IFAC World Congress* (pp. 242–247). Budapest: Pergamon Press. Available at http://www.iwr.uni-heidelberg.de/groups/agbock/FILES/Bock1984.pdf.

Boucekkine, R., Licandro, O., Puch, L., & del Rio, F. (2005). Vintage capital and the dynamics of the AK model. *Journal of Economic Theory*, *120*, 39–72.

Brandt-Pollmann, U., Winkler, R., Sager, S., Moslener, U., & Schlöder, J. (2008). Numerical solution of optimal control problems with constant control delays. *Computational Economics*, *31*, 181–206.

Caulkins, J., Feichtinger, G., Grass, D., Hartl, R., Kort, P., & Seidl, A. (2010a). Two-stage conspicuous consumption model. Working paper.

Caulkins, J., Feichtinger, G., Grass, D., Hartl, R., Kort, P., & Seidl, A. (2011). Optimal pricing of a conspicuous product during a recession that freezes capital markets. *Journal of Economic Dynamics and Control*, *35(1)*, 163–174. Doi:10.1016/j.jedc.2010.09.001.

Caulkins, J., Hartl, R., & Kort, P. (2010b). Delay equivalence in capital accumulation models. *Journal of Mathematical Economics*, *46(6)*, 1243–1246. Doi:10.1016/j.jmateco.2010.08.021.

Collard, F., Licandro, O., & Puch, L. (2008). The short-run dynamics of optimal growth models with delays. *Annales d'Économie et de Statistique*, *90*, 127–143.

Dechert, W. D., & Nishimura, K. (1983). A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory*, *31*, 332–354.

Economist, T. (2008). Riding the rollercoaster: Six firms in cyclical industries and battle excess debt. The Economist.

El-Hodiri, M., Loehman, E., & Whinston, A. (1972). An optimal growth model with time lags. *Econometrica*, *40*, 1137–1146.

Grass, D., Caulkins, J., Feichtinger, G., Tragler, G., & Behrens, D. (2008). *Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror*. Springer, Berlin.

Kalecki, M. (1935). A macroeconomic theory of business cycles. *Econometrica*, *3*, 327–344.

Kolmanovskii, V., & Myshkis, A. (1992). *Applied theory of functional differential equations*. Dordrecht academic publishers.

Kort, P., Caulkins, J., Hartl, R., & Feichtinger, G. (2006). Brand image and brand dilution in the fashion industry. *Automatica*, *42*, 1363–1370.

Leineweber, D. (1999). *Efficient reduced SQP methods for the optimization of chemical processes described by large sparse DAE models* volume 613 of *Fortschritt-Berichte VDI Reihe 3, Verfahrenstechnik*. Düsseldorf: VDI Verlag.

Leineweber, D., Bauer, I., Bock, H., & Schlöder, J. (2003). An efficient multiple shooting based reduced SQP strategy for large-scale dynamic process optimization. Part I: Theoretical aspects. *Computers and Chemical Engineering*, *27*, 157–166.

Nocedal, J., & Wright, S. (2006). *Numerical Optimization*. (2nd ed.). Berlin Heidelberg New York: Springer Verlag. ISBN 0-387-30303-0 (hardcover).

Sethi, S. (1977). Nearest feasible paths in optimal control problems: Theory, examples, and counterexamples. *Journal of Optimization Theory and Applications*, *23*, 563–579.

Sethi, S. (1979). Optimal advertising policy with the contagion model. *Journal of Optimization Theory and Applications*, *29*, 615–627.

Skiba, A. K. (1978). Optimal growth with a convex-concave production function. *Econometrica*, *46*, 527–539.

Times, T. N. Y. (2008). Dim days for luxury hotels. The New York Times. By J. Sharkey.

Winkler, R. (2004). Time-lagged accumulation of stock pollutants. Discussion paper No. 408, Alfred Weber-Institute of Economics, University of Heidelberg.

Winkler, R., Brandt-Pollmann, U., Moslener, U., & Schlöder, J. (2003). Time-lags in capital accumulation. In D. Ahr, R. Fahrion, M. Oswald, & G. Reinelt (Eds.), *Operations Research Proceedings* (pp. 451–458).

Winkler, R., Brandt-Pollmann, U., Moslener, U., & Schlöder, J. (2005). On the transition from instantaneous to time-lagged capital accumulation: The case of leontief type production functions. ZEW Discussion Papers 05-30.