Numerical solution of optimal control problems with constant control delays

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Abstract: We investigate a class of optimal control problems that exhibit constant exogenously given delays in the control in the equation of motion of the differential states. Therefore, we formulate an exemplary optimal control problem with one stock and one control variable and review some analytic properties of an optimal solution. However, analytical considerations are quite limited in case of delayed optimal control problems. In order to overcome these limits, we reformulate the problem and apply direct numerical methods to calculate approximate solutions that give a better understanding of this class of optimization problems.

In particular, we present two possibilities to reformulate the *delayed* optimal control problem into an *instantaneous* optimal control problem and show how these can be solved numerically with a state-of-the-art direct method by applying Bock's direct multiple shooting algorithm. We further demonstrate the strength of our approach by two economic examples.

Keywords: delayed differential equations, delayed optimal control, numerical optimization, time-to-build

JEL-Classification: C63, C61

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1 Introduction

Many intertemporal economic applications exhibit the mathematical form of optimal control problems, where an objective function (e.g., intertemporal welfare, profit, costs, etc.) is sought to be maximized or minimized subject to a system of equations of motion, which determine the interaction of the stock and the control variables. Recently, economists consider increasingly models, where economic systems do not react *instantly* but with a *delay* to changes in external influences (e.g., investment-lags, transportation-lags, lags in habit formation, etc.).

One way to deal with such a delayed structure in continuous time is the use of delayed differential equations. However, the use of delayed differential equations in optimal control frameworks exhibits severe analytical and numerical difficulties. In general, even the linear approximation of the system dynamics around the stationary state is governed by a system of differential-difference equations of neutral type, which is, in general, *not* analytically solvable. As a consequence, numerical optimization methods play an important role in analyzing and understanding the behavior of delayed optimal control problems.

In this paper we show how optimal control problems in continuous time with one stock and one control variable with a constant time delay can be solved numerically. We reformulate the original problem in two different ways into constrained control problems in ordinary differential equations with higher dimensional control functions respectively state variables. Thus, we avoid the solution of the delayed system at the cost of higher dimensionality. Furthermore, we show how to solve the reformulated control problems by *Bock's direct multiple shooting method.* The power of the solution method is demonstrated by treating two typical economic examples. Furthermore, we discuss how our framework relates to the following different classes of economic problems discussed in the literature.

One strand of economic literature, where delayed structures play a crucial role are *investment gestation lags.* Following a denotation, which is, for example, used by Altuǧ (1993) and Peeters (1996), one can further distinguish between *delivery lags* (i.e., investment for new capital goods is made at time t but the new capital goods need some time σ to be delivered and, thus, to be used productively), and *time-to-build* (i.e., capital goods need some time σ over which they require investments in their production). To the best of our knowledge, El-Hodiri *et al.* (1972), who derive a generalized maximum principle for a growth model with heterogeneous capital goods and exogenously given and constant delays between control and state variables, is the only contribution of the former class applying delayed differential equations in a continuous time framework. However, more recent model specifications in discrete time include, for example, Altuǧ (1993) and Peeters (1996).

The term time-to-build was coined by Kydland & Prescott (1982) who, following an idea first posed by Kalecki (1935), empirically analyzed in how far time-to-build could explain real business cycles observed in reality. Rustichini (1989) and Asea & Zak (1999) showed in simple delayed continuous time optimal control models with one capital good (but a different lag structure) that the time-to-build feature is the driving force for the

oscillatory system dynamics.[1]

Another strand of the literature where delayed differential equations were successfully applied are vintage capital growth models. In vintage capital models, capital of different age may exhibit different productivity due to technical progress and/or effects of non-exponential depreciation. The general problem in vintage capital models is keeping track of the capital goods of different ages, which can be formalized by using delayed differential equations. Benhabib & Rustichini (1991), Boucekkine *et al.* (1997a) and Boucekkine *et al.* (1997b) assume linear utility and, thus, avoid the problem of functional differential equations of neutral type. This assumption is relaxed in Boucekkine *et al.* (1998, 2001) and Boucekkine *et al.* (2005). While Boucekkine *et al.* (1998, 2001) rather concentrate on the numerical solution of specific vintage capital specifications, Boucekkine *et al.* (2005) characterize analytically the complete dynamics of a simple AK vintage capital model with constant lifetime of the capital good (i.e., one-hoss shay depreciation). Analogously to age structures in physical capital, one can explicitly consider age structures in human capital, which are generated by endogenous schooling and retirement decisions of the economic agents. De la Croix & Licandro (1999), Boucekkine *et al.* (2002) and Boucekkine *et al.* (2004) investigate age structures in delayed continuous time optimal control problems.

In addition, delayed optimal control in continuous time can contribute to strands of economic literature, where it has not been applied so far. As an example think of *habit formation*, where time lags also play a crucial role. With habit formation, utility depends not only on current outcomes but also on a stock of habits, which is in general some weighted average of previous outcomes (e.g., Boyer 1978, Carroll *et al.* 2000). Although delayed differential equations have, to the best of our knowledge, not been used so far in the economic literature to investigate habit formation,[2] we briefly discuss this issue in section 5.

Other potential applications are in the field of environmental economics, where damages from stocks of pollution are considered. Often these stocks do not instantaneously accumulate to the emission of the pollutants but need some time due to transportation processes. Prime examples include ground water contamination by excessive fertilizing and the destruction of the ozone layer by the emission of CFCs. One of our examples in section 4 refers to the CFC case.

The remainder of the paper is structured as follows. Section 2 defines the class of delayed optimal control problems we seek to solve numerically. Furthermore, we review some qualitative properties of the optimal path and outline the difficulties for numerical solution methods. In section 3 we reformulate the optimal control problem in a suitable way to allow an application of the direct multiple shooting method. Two examples demonstrate the range of application for the solution method in section 4. In section 5, we discuss the robustness of our approach to changes in the model specifications and show how our approach can be applied to different classes of economic problems. Finally,

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[1] However, we will argue in section 5 that in the denotation of Altuğ (1993) and Peeters (1996) their formulation is rather of the delivery lag than the time-to-build type.

[2] In Collard *et al.* (2004) it is mentioned as an example for the application of delayed differential equations in economic optimization models but not further investigated.

section 6 concludes.

2 A generic optimal control problem with delayed equation of motion

We investigate a class of optimal control problems with one stock and one control variable and a control-delayed equation of motion of the stock variable. As usual in economic applications, we consider the maximization of an objective functional W, which is the discounted infinite integral over an autonomous felicity function F. With a stock variable x and a control variable u, the optimal control problem reads

$$\max_{x(t),u(t)} W = \int_0^\infty F(x(t), u(t)) \exp[-\rho t] \, dt \qquad (1a)$$

subject to

$$\dot{x}(t) = u(t-\sigma) - \gamma x(t) \ , \qquad (1b)$$

$$u(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \ , \qquad (1c)$$

$$x(0) = x_0 \ , \qquad (1d)$$

$$u(t) = \xi(t) \ , \quad t \in [-\sigma, 0) \ , \qquad (1e)$$

where ρ denotes the constant and positive discount rate, σ is a constant delay or time-lag, and γ is a constant decay rate. In addition, F is assumed to be twice continuously differentiable with respect to both arguments.

The crucial feature is that the control $u(\cdot)$ enters with a delay σ as $u(t - \sigma)$ in constraint (1b), while it is evaluated at time t as $u(t)$ in the objective functional (1a). In general, a differential equation with a delay in the state variables or control functions is referred to as a *delayed differential-difference equation* (DDE). Other common terms are *retarded linear functional differential equation* or *differential-difference equation of retarded type*. For an introduction to DDEs see Asea & Zak (1999: section 2) and Gandolfo (1996: chapter 27). A detailed exposition of (linear) functional differential equations is given in Bellman & Cooke (1963), Driver (1977), Hale (1977), Kolmanovskii & Nosov (1986) and Kolmanovskii & Myshkis (1999).

In contrast to models with instantaneous equations of motion, besides an initial value x_0 for the stock x, also an initial path ξ for the control $u(\cdot)$ in the time interval $[-\sigma, 0)$ has to be specified (or also optimized). Note that the path of the stock x in the time interval $t \in [0, \sigma]$ is completely determined by the initial stock x_0, the initial path $\xi(\cdot)$, and the retarded equation of motion in (1). Thus, optimal control problems which are governed by a retarded equation of motion exhibit an additional moment of inertia, as the variation of the stock reacts with a delay to the control. Although the equation of motion is very specific, the maximization problem (1) represents numerous economic models as we outline by two examples in section 4 and discuss further in section 5.

Given that the felicity function F is strictly concave and the restrictions (1c) on the control u are not binding, one obtains the following system of differential equations for

an optimal solution from the necessary conditions and the equation of motion for the stock x (1b):

$$
\begin{aligned}
\dot{u}(t) &= \frac{F_u(t)}{F_{uu}(t)}(\gamma + \rho) + \frac{F_x(t+\sigma)}{F_{uu}(t)}\exp[-\rho\sigma] + \frac{F_{xu}(t)}{F_{xx}(t)}(\gamma x(t) - u(t-\sigma)) , \\
\dot{x}(t) &= u(t-\sigma) - \gamma x(t) .
\end{aligned}
\tag{2}
$$

Note that $\dot{u}$ and $\dot{x}$ also depend on *advanced* (i.e., at a later time) and on *retarded* (i.e., at an earlier time) variables. Hence, (2) forms a system of *functional differential equations of neutral type*. Obviously, a possible approach to numerically solve the optimization problem (1) is to numerically solve the system of functional differential equations (2). However, recall that the system (2) is only the solution of the original optimization problem (1) in the case of an interior solution. Moreover, to determine a unique solution for (2), additional information about the first derivatives $\dot{x}$ and $\dot{u}$ at some point t is needed a priori. Therefore, we shall introduce a direct approach in this paper to numerically solve the original control problem (1) directly.

Before we show how to reformulate the optimization problem (1) in order to derive a numerical solution, we briefly recall some of its analytical properties, which are derived in detail in Winkler (2004).

First, the stationary state $(x^\star, u^\star)$ of the system of functional differential equations (2), which can be be shown to exist and is also unique if the felicity function F satisfies Inada conditions, is given by the following (implicit) equations:

$$
\begin{aligned}
-\frac{F_x(x^\star, u^\star)}{F_u(x^\star, u^\star)} &= (\gamma + \rho)\exp[\rho\sigma] , \\
u^\star &= \gamma x^\star .
\end{aligned}
\tag{3}
$$

Second, linearizing the system of functional differential equations (2) around the stationary state $(x^\star, u^\star)$ yields a *quasi-polynomial* as characteristic equation, which has in general an infinite number of (complex) roots. However, the characteristic equation reduces to a simple quadratic equation with one positive and one negative real characteristic root for the special case that the partial derivative $F_{xu}(x^\star, u^\star) = 0$.

Although the characteristic roots are not analytically solvable, the characteristic equation can be shown to exhibit an infinite number of complex solutions with positive real parts and an infinite number of complex solutions with negative real parts. As a consequence, the stationary state $(x^\star, u^\star)$ is a saddle point and, thus, for all initial stocks x_0 and all initial control paths ξ, there exists a unique optimal path which converges asymptotically towards the stationary state.[3]

In summary, we have monotonic convergence if the felicity function F is additively separable, otherwise oscillations may occur.

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[3] If the characteristic equation exhibits purely imaginary roots (i.e., complex roots with vanishing real parts), the system dynamics may exhibit so called *limit-cycles*. That is, the optimal paths oscillate around the stationary state without converging towards or diverging from it. Limit-cycles in the case of delayed optimal control problems have been discussed by Rustichini (1989) and Asea & Zak (1999).

3 Numerical solution of the optimal control problem

Despite the analytical derivation of the qualitative properties of the optimal path, even the linearized approximation around the stationary state of the system of functional differential equations (2) is *not* analytically solvable. As a consequence, numerical optimization methods play an important role to analyze and understand the behavior of delayed optimal control problems. In the following section we show two ways how to reformulate the original problem in order to make it tractable for *Bock's direct multiple shooting method*, a highly efficient algorithm for the numerical solution of constrained optimal control problems in ordinary differential equations (ODE) and differential-algebraic equations (DAE).

3.1 Reformulation of the delayed optimal control problem

First, we have to restrict the time horizon for the numerical optimization to a finite value t_f, a caveat every numerical algorithm has to deal with. This poses no major problems as, according to the stability properties of the optimal solution outlined in the previous section, the results will be arbitrarily close to the problem with an infinite time horizon if t_f is sufficiently large. As we shall see, it is most convenient to set t_f to be a (large) multiple of the time-lag σ. In the delayed control problem (1), the delay σ solely appears in the control variable in the equation of motion (1b). Hence, it is possible to reformulate this delayed optimal control problem with one state variable into an instantaneous optimal control problem with several state variables. Thus, we can avoid to explicitly numerically treat the time-lag at the cost of higher dimensionality.[4]

To see this, we split the time horizon t_f into n parts each the length of the delay σ and formulate the equation of motion separately in each of the resulting intervals. Thus, we obtain for the first interval $t \in [0, \sigma)$

$$\dot{x}(t) = \xi(t-\sigma) - \gamma x(t) , \qquad t \in [0, \sigma) , \tag{4}$$

where ξ is the initial control path in the time interval $t \in [-\sigma, 0)$. In the second interval $t \in [\sigma, 2\sigma)$ the equation of motion yields

$$\dot{x}(t) = u(t-\sigma) - \gamma x(t) , \qquad t \in [\sigma, 2\sigma) , \tag{5}$$

and so on.

The clue is to interpret each of the resulting DDEs as an independent differential equation. By introducing n new stock variables x_l and $n-1$ new control variables u_l with

$$x_l(t) = x(t+(l-1)\sigma) , \quad u_l(t) = u(t+(l-1)\sigma) , \quad t \in [0, \sigma) , \tag{6}$$

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[4] This method is a straightforward generalization of the well-known method of steps in Bellman & Cooke (1963) to solve delayed differential-difference equations. The method of steps is also applied in Boucekkine *et al.* (1997a) to numerically solve a system of delayed differential-difference equations.

we achieve the following system of ordinary differential equations:

$$
\begin{aligned}
\dot{x}_1(t) &= \xi(t-\sigma) - \gamma x_1(t) , & t \in [0, \sigma) , \\
\dot{x}_2(t) &= u_1(t) - \gamma x_2(t) , & t \in [0, \sigma) , \\
&\;\;\vdots \\
\dot{x}_{n-1}(t) &= u_{n-2}(t) - \gamma x_{n-1}(t) , & t \in [0, \sigma) , \\
\dot{x}_n(t) &= u_{n-1}(t) - \gamma x_n(t) , & t \in [0, \sigma) ,
\end{aligned}
\tag{7}
$$

Thus, we can reformulate the original optimization problem (1) as:

$$
\max_{x_l(t), u_l(t)} \int_0^\sigma \sum_{l=1}^n F(x_l(t), u_l(t)) \exp[-\rho(t + \sigma(l-1))] \, dt
\tag{8a}
$$

subject to

$$
\begin{aligned}
\dot{x}_1(t) &= \xi(t-\sigma) - \gamma x_1(t) , \\
&\;\;\vdots \\
\dot{x}_n(t) &= u_{n-1}(t) - \gamma x_n(t) ,
\end{aligned}
\tag{8b}
$$

and the restrictions for the control variables u_l:

$$
u_l(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} .
\tag{8c}
$$

Furthermore we have to introduce additional coupled boundary conditions for the stock variables x_l at time $t = 0$ and $t = \sigma$ to ensure the continuity of the stock variable x of the original problem:

$$
x_l(\sigma) = x_{l+1}(0) , \quad l = 1, \ldots, n-1 .
\tag{8d}
$$

Finally, the condition (1d) for the initial stock x_0 translates into

$$
x_1(0) = x_0 .
\tag{8e}
$$

Note that we need only to determine $n-1$ control paths in the interval $[0, \sigma]$ as the optimal path for the stock in the interval $t \in [(n-1)\sigma, n\sigma)$ is completely determined by the stock at $t = (n-1)\sigma$, $x_{n-1}(\sigma)$, the control $u_{n-1}(t)$ and the equation of motion.

Remark 1. In addition to transforming the retarded optimization problem in a suitable form for numerical solution methods, the reformulation (8) also gives an intuitive explanation why the optimal control problem (1)

(i) exhibits an infinite number of characteristic roots in general, and

(ii) exhibits only two characteristic roots in the case that the felicity function F is additively separable.

To see (i), recall that the characteristic equation for an optimal control problem with n stock variables is a polynomial of order $2n$, which has in general $2n$ characteristic roots (although it may be less than $2n$ distinct roots as there may be multiple roots). Independent of the time-lag σ, n tends to infinity if we extend the time horizon $t_f \to \infty$. Thus, for an infinite time horizon t_f, the retarded optimization problem (1) with one stock variable is equivalent to an ordinary optimal control with an infinite number of stock variables, resulting in a characteristic equation with an infinite number of characteristic roots.

To see (ii), recall that F is additively separable is equivalent to $F(x,u) = G(x) + H(u)$. Thus, the objective functional (8a) yields for an infinite time horizon

$$\max_{x_l(t), u_l(t)} \int_0^\sigma \sum_{l=1}^\infty [G(x_l(t)) + H(u_l(t))] \exp[-\rho(t + \sigma(l-1))] \, dt \ . \tag{9}$$

$G(x_1(t))$ is independent of variations in the control variables u_l, $l \geq 1$, as it is completely determined by the initial path ξ, the initial stock x_0 and the equation of motion. Therefore, it is sufficient to maximize the objective functional without the term exhibiting $G(x_1(t))$. Hence, we can rearrange the remaining terms to yield:

$$\max_{x_l(t), u_l(t)} \int_0^\sigma \sum_{l=2}^\infty [G(x_l(t)) + H(u_{l-1}(t)) \exp[\rho\sigma]] \exp[-\rho(t + \sigma(l-1))] \, dt \tag{10}$$

Transforming the objective function back to one stock and one control variable yields:

$$\max_{x(t), u(t)} \int_0^\infty [G(x(t+\sigma)) \exp[-\rho\sigma] + H(u(t))] \exp[-\rho t] \, dt \tag{11}$$

Introducing a new stock variable $\hat{x}(t) = x(t+\sigma)$ we achieve the following ordinary optimal control problem:

$$\max_{x(t), u(t)} \int_0^\infty [G(\hat{x}(t)) \exp[-\rho\sigma] + H(u(t))] \exp[-\rho t] \, dt \tag{12a}$$

subject to

$$\dot{\hat{x}}(t) \;=\; u(t) - \gamma\hat{x}(t) \ , \tag{12b}$$
$$u(t) \;\in\; [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \ , \tag{12c}$$
$$\hat{x}(0) \;=\; x_\sigma \ , \tag{12d}$$

where x_σ is the value of the original stock variable x at time σ (which is completely determined by x_0, ξ and the original equation of motion). Thus, the retarded optimal control problem (1) is formally equivalent to the ordinary optimal control problem (12) with one stock and one control variable. As a consequence, its characteristic equation is a polynomial of second order, which is known to exhibit two characteristic roots.

Remark 2. Despite the intuitive explanation for the qualitative system dynamics in the general case and in the case of an additively separable felicity function F, the reformulation (8) does not promote the analytical derivation of the optimal solution in the general case. This holds as the additional coupled boundary constraints (8d), which guarantee the continuity of the original stock variable x, pose severe obstacles for an analytical solution.

Problem (8) is useful for analytical considerations as outlined in Remark 1 and can be solved by the direct multiple shooting method as will be shown in section 3.2. However, for a given time horizon t_f, the number n of differential state and control functions becomes quite large for small values of the time-lag σ. Therefore, we also consider another reformulation of the problem (1) with fixed dimension of state and controls.

To this end we introduce an additional control function. While $u_2(t)$ is the same as $u(t)$ before and denotes the control at time t, $u_1(t)$ represents the retarded control $u(t-\sigma)$. Thus, u_1 and u_2 are coupled by $u_1(t) = u_2(t-\sigma)$ for $t \geq \sigma$ and $u_1(t) = \xi(t)$ for $0 \leq t \leq \sigma$. Then, problem (1) is equivalent to

$$\max_{x(t),u_1(t),u_2(t)} \int_0^\infty F(x(t), u_2(t)) \exp[-\rho t]\, dt \tag{13a}$$

subject to

$$\dot{x}(t) = u_1(t) - \gamma x(t) \tag{13b}$$

$$u_1(t), u_2(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \tag{13c}$$

$$x(0) = x_0\,, \tag{13d}$$

$$u_1(t) = \xi(t - \sigma), \quad 0 \leq t < \sigma, \tag{13e}$$

$$u_1(t) = u_2(t - \sigma), \quad t \geq \sigma. \tag{13f}$$

Problem (13) still contains a retarded term, but it has moved from the differential equation (13b) to a constraint on the controls (13f), that can be dealt with efficiently by the direct multiple shooting method. In contrast to the reformulation (8), only one additional control variable has been introduced independently of the time horizon t_f and the time-lag σ.

3.2 Bock's direct multiple shooting method

In order to solve the reformulated optimal control problems (8) and (13) numerically, we apply the *direct multiple shooting* method originally developed by Bock and his coworker Plitt (1981), Bock & Plitt (1984). Let us consider an optimal control problem of the form

$$\max_{x(t),u(t)} \int_{t_0}^{t_f} L(x(t), u(t))\, dt \tag{14a}$$

subject to

$$\dot{x}(t) \quad = \quad f(x(t), u(t)), \quad t \in [t_0, t_f], \tag{14b}$$
$$0 \quad \leq \quad c(x(t), u(t)), \quad t \in [t_0, t_f], \tag{14c}$$
$$0 \quad = \quad r^{\text{eq}}(x(\tau_0), x(\tau_1), \dots, x(\tau_m)), \tag{14d}$$
$$0 \quad \leq \quad r^{\text{ieq}}(x(\tau_0), x(\tau_1), \dots, x(\tau_m)), \tag{14e}$$

with all occurring functions twice differentiable.

We approximate the n_u-dimensional control function $u(\cdot)$ by functions with local support and finitely many parameters. To this end we introduce a time grid

$$t_0 = \tau_0 < \tau_1 < \dots < \tau_m = t_f \tag{15}$$

and split the time horizon $[t_0, t_f]$ into m so called *multiple shooting intervals* $[\tau_{j-1}, \tau_j]$, where $j = 1, \dots, m$. On each multiple shooting interval we define a typically low dimensional control parameterization, e.g., a linear approximation $\phi^j(t)$ of the controls $u(t)$ by

$$\phi^j(t) := q_1^j + q_2^j t , \quad t \in [\tau_{j-1}, \tau_j] , \tag{16}$$

with vector valued parameters q^j.

We introduce m variables $s^j \in \mathbb{R}^{n_x}$ as initial values for the differential states on each multiple shooting interval $[\tau_{j-1}, \tau_j]$. The ODE (14b) is solved independently on every interval with initial values

$$x(\tau_j) = s^j , \quad j = 0, \dots, m-1 . \tag{17}$$

To ensure continuous state trajectories $x(\cdot)$, the values at the end of interval j, obtained by integration with initial value s^j and control parameter q^j, have to coincide with the initial state vector of the next interval $j + 1$:

$$x(\tau_{j+1}; s^j, q^j) = s^{j+1} , \quad j = 0, \dots, m-1 . \tag{18}$$

These so-called *matching conditions* (18) allow to eliminate the additional degrees of freedom introduced by the supplementary optimization parameters s^j by *condensing* (for details see Bock & Plitt 1984). Note that the conditions (18) are required to be satisfied only at the final solution of the problem and not during intermediate iterations of the optimization algorithm. Therefore, the direct multiple shooting method is also referred to as an *all-at-once-approach*, solving the simulation and optimization task at the same time. This allows to incorporate expert knowledge about the trajectory behavior into the initial values of the state trajectory and typically leads to good convergence properties of the method. The path and control constraints (14c) have to hold on the whole time interval $[t_0, t_f]$. To deal with this numerically, in the direct multiple shooting method these constraints are formulated as point constraints on a suitable finite time grid.

Following these lines, problem (14) is now an optimization problem in the variables q^j and s^j. It contains equality constraints that stem from the interior point equality

constraints (14d) and the matching conditions (18), and inequality constraints that stem from the interior point equality constraints (14e) and the discretized path constraints (14c).

Subsuming all variables s^j and q^j into $w \in \mathbb{R}^{n_w}$ and rewriting the objective function as well as the constraints in adequate functions F, G and H, we obtain a non-linear program (NLP)

$$\min_{w} F(w) \quad \text{subject to} \quad \begin{cases} G(w) &=& 0 \\ H(w) &\geq& 0 \end{cases}, \tag{19}$$

that can be solved by tailored methods. For example, by sequential quadratic programming (SQP) in combination with an efficient evaluation of all occurring functions, and the generation of derivatives, for example, by internal numerical differentiation. See Leineweber *et al.* (2003) for details and further references.

Now, let us consider an application of the direct multiple shooting method to the reformulations (8) and (13) of the original problem (1). Obviously, (8) is of the form (14) and can, thus, be solved with the direct multiple shooting method as described above. However, reformulation (13) contains an additional constraint (13f), which is not contained in the standard problem formulation (14).

Here, the approximation of the control functions allows to guarantee (13f) – if the corresponding entries of $u_1(t)$ in q^j and the ones of $u_2(t)$ in q^{j-1} match at all times τ_j, then the equation holds on the whole time horizon (as each piecewise linear control is uniquely determined by two points). If we extend the interior point equality constraint (14d) to allow also for arguments $u(\tau_j)$ (which is typically omitted, as only measurable influence of a control function shall be considered), then the direct multiple shooting method can be applied to solve both problems (8) and (13).

4 Examples

In the following we illustrate the potential of the numerical solution method described in the previous section by two examples, which stem from our research on delayed optimal control problems. The first example shows how numerical optimization can be used to analyze the transition from instantaneous to delayed stock accumulation. The second example focuses on the influence of the initial path ξ on the optimal paths of a delayed optimal control problem.

4.1 The transition from instantaneous to delayed capital accumulation

The first example is an optimal control capital accumulation model with an investment gestation lag. In fact, we consider a delivery lag, i.e., an exogenously given delay between investment and capital accumulation, which is discussed in detail in Winkler *et al.* (2005).

Consider an economy with one non-producible input factor, for example, labor, which is given in constant amount $\bar{l}$ and distributed to three linear-limitational production processes. The first process produces one unit of the consumption good with one unit

of labor. The second process combines λ units of labor together with κ units of capital to produce one unit of the consumption good. The third process creates one unit of investment from one unit of labor. Thus, we derive

$$c_1(t) = l_1(t) \, , \tag{20}$$

$$c_2(t) = \min\left[\frac{l_2(t)}{\lambda}, \frac{k(t)}{\kappa}\right] \, , \tag{21}$$

$$i(t) = l_3(t) \, , \tag{22}$$

where l_i denote the amount of labor employed in process i ($i = 1, 2, 3$). Assuming efficient production (i.e., $l_2(t)/\lambda = k(t)/\kappa$), and that the labor restriction holds with equality (i.e., $\sum_i l_i(t) = \bar{l} \; \forall t$), total consumption $c(t) = c_1(t) + c_2(t)$ yields:

$$c(t) = \bar{l} + \frac{1-\lambda}{\kappa} k(t) - i(t) \, . \tag{23}$$

Further, we assume that investment at time t increases the capital stock k delayed at time $t+\sigma$, and that the capital stock deteriorates at the positive and constant rate γ

$$\dot{k}(t) = i(t-\sigma) - \gamma k(t) \, . \tag{24}$$

In addition, we assume that the capital stock k cannot be consumed (i.e., $i(t) \geq 0$). Assuming that the objective is to maximize intertemporal welfare, which is the discounted infinite integral of instantaneous welfare $V(c(t))$, the optimal control problem reads:

$$\max_{k(t),i(t)} \int_0^\infty V\left(\bar{l} + \frac{1-\lambda}{\kappa} k(t) - i(t)\right) \exp[-\rho t] \, dt \tag{25a}$$

subject to

$$\dot{k}(t) = i(t-\sigma) - \gamma k(t) \, , \tag{25b}$$

$$i(t) \geq 0 \, , \tag{25c}$$

$$\bar{l} - \frac{\lambda}{\kappa} k(t) - i(t) = c(t) - \frac{1}{\kappa} k(t) \geq 0 \, , \tag{25d}$$

$$i(t) = \xi(t) = 0 \, , \quad t \in [-\sigma, 0) \, , \tag{25e}$$

$$k(0) = 0 \, . \tag{25f}$$

The restriction (25d) ensures that $c_1 \geq 0$. When it is binding, all labor is used to employ and maintain the capital stock. This implies that the consumption good is exclusively produced by the capital intensive process (21). For the following calculations we choose $V(c(t)) = \ln c(t)$, $\bar{l} = 26\frac{2}{3}$, $\lambda = 0.8$, $\kappa = 0.3$, $\gamma = 0.15$, $\rho = 0.1$, $t_f = 60$, $k_0 = 0$ and the initial path $\xi(\cdot) \equiv 0$.

The resulting optimization problem (25) is almost equivalent to the problem (1) discussed in section 2. As the additional inequality constraint (25d) fits directly into the definition of path and control constraints (14c), both reformulations (8) and (13) of (25) can be solved by the direct multiple shooting method.

Delay σ	(8) dense			(8) sparse			(13)		
	n_w	iters	time	n_w	iters	time	n_w	iters	time
0.5	605	47	208	605	47	110	724	20	10
0.4	755	50	419	755	50	224	904	23	24
0.3	1005	50	1094	1005	50	521	1204	23	53
0.2	1505	—	—	1505	—	—	1804	23	287
0.1	3005	—	—	3005	—	—	3604	14	1331

Tabelle 1: Comparison of the number of variables n_w of the resulting NLP, number of SQP iterations and computing time in seconds needed to reach a KKT tolerance of 10^{-6}.

Action	(8) dense		(8) sparse		(13)	
	time	percent	time	percent	time	percent
Sensitivity generation	122	60.4%	30.0	26.7%	2.2	9.9%
State integration	0.7	0.3%	0.5	0.4%	0.8	4.1%
Condensing	3.2	1.6%	3.3	3.0%	8.8	39.8%
Solution of QPs	74.4	36.8%	74.5	68.5%	7.6	35.5%
Rest	1.76	0.9%	1.6	1.4%	2.3	10.5%

Tabelle 2: A typical distribution of computing times. The absolute times given in seconds have been scaled to be independent of the number of iterations.

Whereas the optimal solutions of the two different reformulations are, of course, identical, they exhibit different computational performance. Table 1 shows a comparison between the two approaches. All computations have been performed with the state-of-the-art optimal control software package MUSCOD-II, see Leineweber (1999), on a Pentium notebook with 1.5 GHz. Note that for the calculations the underlying control discretization grid has been chosen identical to the equidistant grid with distance σ. The computation times are given in seconds and describe how long it took before an accuracy of 10^{-6} of the Karush-Kuhn-Tucker (KKT) conditions was achieved. Obviously, problem reformulation (13) is much more suited for small time lags σ. The number of variables n_w of the non-linear program (NLP) is not the crucial indicator, though, as can be seen in table 1. Let us investigate in more detail what happens. Table 2 shows the distribution of the computing times for specific tasks. The times spent on condensing, online graphics, constraint reductions and other calculations are more or less the same. Also the time spent on state integration is compared to the rest.

The main difference is in the required time for calculating derivative information by internal numerical differentiation and the solution of the condensed quadratic programs (QPs). The size of the Jacobian matrix needed to calculate the sensitivities depends on the number of variables and is, thus, much higher for (8) than for (13). This effect can be reduced by a factor of about four by exploiting sparsity[5] (compare middle column in tables 1 and 2) with an advanced solver such as DAESOL (see Bauer 1999), but there is

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[5] A matrix is called *sparse* if it contains only few nonzero entries, otherwise it is called *dense*.

still a considerable difference to the formulation (13) with only one state and two control variables.

The solution of the QPs in the SQP scheme is also much more expensive for problem (8), as condensing does not reduce the number of variables actually given to the QP. If we do not perform condensing for problem (13), the computing time for "Solution of QPs" goes up to 68 seconds and almost reaches the level of problem (8).

To sum up, reformulation (13) is better suited for numerical calculations than (8), as it has a structure that can be better exploited by standard direct multiple shooting methods. Hence, in the following we will only use this formulation for our calculations.

We now solve the model to investigate the system dynamics dependent on the time-lag σ. In particular, we analyze the transition between instantaneous and delayed capital accumulation by solving (25) respectively (13) for different time-lags σ. Figure 1 shows optimized paths for time-lags σ ranging from 0 to 0.5. Consistent with the findings in section 2 the optimal paths converge monotonically towards the stationary state for $\sigma = 0$ and oscillatory and exponentially damped for $\sigma > 0$.

The continuous transition from monotonic to increasingly oscillatory optimal paths for increasing time-lags σ can be seen in figure 2. The exogenous parameters are identical to the calculations for figure 1. The interval for the time-lag $\sigma \in [0.1, 0.5]$ has been split into a grid of 400 equidistant points. For each of these σs the optimal control problem has been solved and the resulting graphs have been composed to the 3-dimensional plots in figure 2. They show how the optimal paths evolve from monotonic to oscillatory paths for increasing time-lag σ.

4.2 The influence of the initial path on the optimal control of delayed pollution stock accumulation

The second model, first introduced in Winkler (2004), discusses the case of delayed pollution accumulation. The idea is that a joint output of production, which is released into the environment, accumulates there to a pollutant stock, which exhibits a negative effect on the economy. Although the following model has been inspired by the environmental problem of the emission of chlorofluorocarbons (CFCs), it is applicable to various stock pollutants. CFCs are a prime example of delayed accumulating stock pollutants. They have been widely used as cooling agents in refrigeration and air conditioning, as propellants in aerosols sprays and foamed plastics, and as solvents for organic matters and compounds. The CFCs have been valued because of their favorable chemical and biological characteristics. They are chemically inert, not inflammable and non-toxic. Unfortunately, in the stratosphere the CFCs cause the depletion of the ozone layer, which shields the earth's surface from ultraviolet radiation. Once released, the CFCs need 5–10 years to reach a height of about 30 km, where the depletion of the ozone layer starts. Hence, the stock of stratospheric CFCs reacts to the emissions of CFCs with a delay of 5–10 years.

Consider an economy with one non-producible input of production, for example, labor, which is given in a constant maximal amount $\bar{l}$ and distributed among two production processes in the economy. The first production process produces a consumption good c
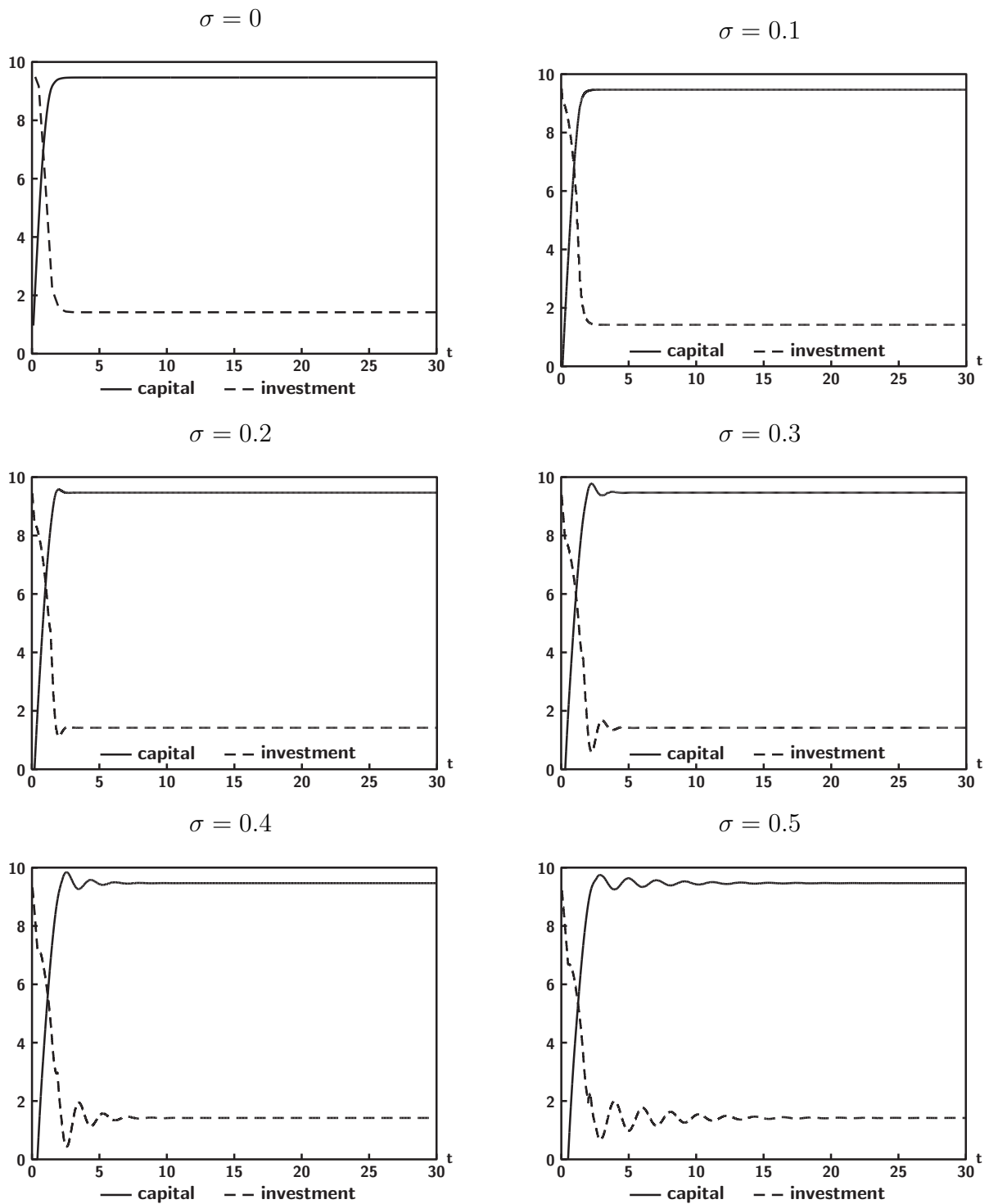
Abbildung 1: Optimal paths for capital and investment for selected time-lags $\sigma \in [0, 0.5]$ between investment and capital accumulation.
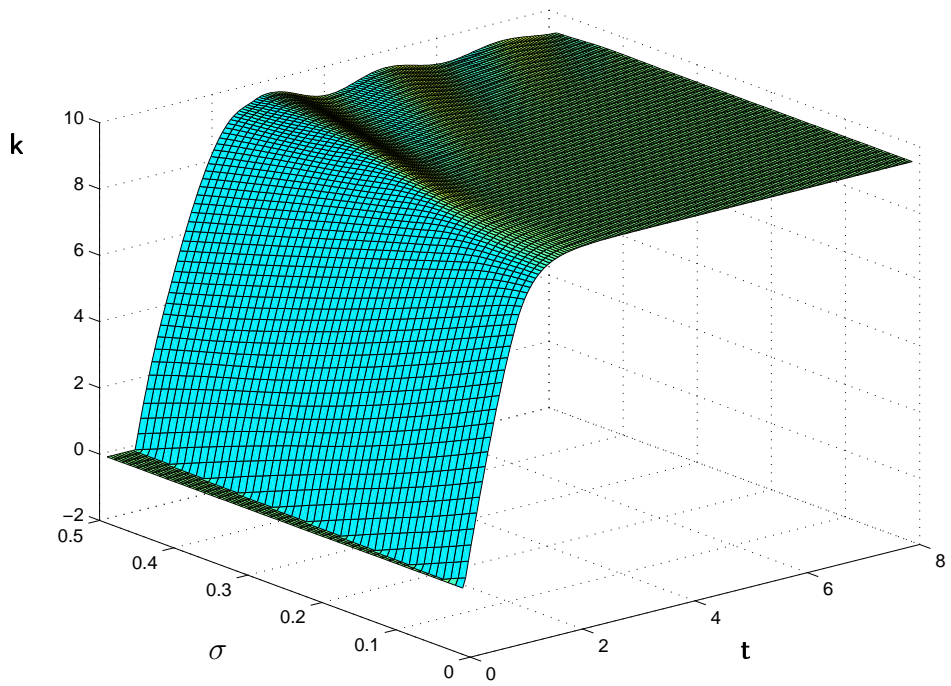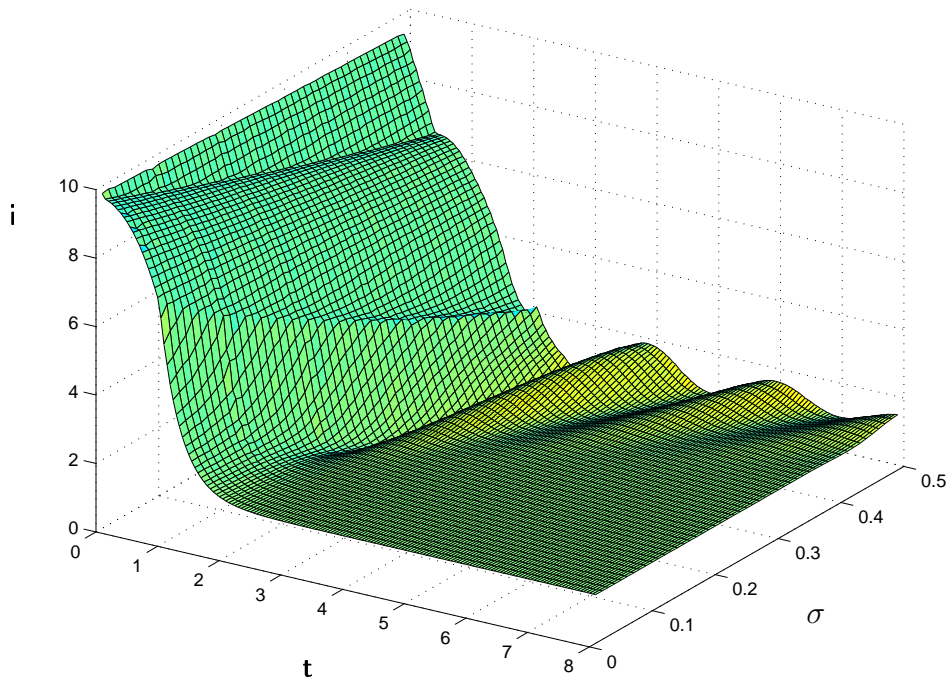
Abbildung 2: Optimal investment (top) and capital (bottom) paths for time-lags $\sigma \in [0, 0.5]$ between investment and capital accumulation. The third axis denotes increasing time-lags σ.

with constant returns to labor

$$c(t) = l_1(t) \ , \tag{26}$$

where l_1 denotes the amount of labor employed to the consumption good production. In addition, the production of each unit of consumption good gives rise to one unit of gross emissions e^{gross}:

$$e^{gross}(t) = c(t) = l_1(t) \ . \tag{27}$$

The second production process is an abatement process, which reduces net emissions e

$$e(t) = e^{gross}(t) - a(t) \ , \tag{28}$$

where a denotes the amount of emissions abated. Denoting the amount of labor employed to the abatement process by l_2, the amount of abated emissions is given by:

$$a(t) = \sqrt{\alpha l_2(t)} \ , \quad \alpha > 0 \ . \tag{29}$$

The net emissions e are considered to accumulate the pollution stock s with a time-lag σ. In addition, the pollution stock s decays at a constant rate γ

$$\dot{s}(t) = e(t-\sigma) - \gamma s(t) \ . \tag{30}$$

The stock of pollutant s exhibits a negative external effect on the economy, as it reduces the effective labor force l:

$$l(t) = \bar{l} - \beta s(t)^2 \ , \quad \beta > 0 \ . \tag{31}$$

In the case of CFCs, one might think of an increase in the rate of skin cancer with increasing stock of the pollutant, which prevents increasingly more people from working. Note that the pollution stock s exhibits increasing marginal damage. Given efficient production (i.e., the labor constraint holds with equality $l(t) = l_1(t) + l_2(t)$), consumption is given by

$$c(t) = c(e(t), s(t)) = \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\left(\lambda - \beta s(t)^2 - e(t)\right) + \alpha^2} \right] \ . \tag{32}$$

Again, we assume that the objective is to maximize intertemporal welfare, which is the discounted infinite integral of instantaneous welfare $V(c(t))$. Thus, the optimal control problem reads:

$$\max_{s(t),e(t)} \int_0^\infty V\left(\frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\left(\lambda - \beta s(t)^2 - e(t)\right) + \alpha^2} \right] \right) \exp[-\rho t] \, dt \tag{33a}$$

subject to

$$\dot{s}(t) = e(t-\sigma) - \gamma s(t) \ , \tag{33b}$$
$$e(t) = \xi(t) \ , \quad t \in [-\sigma, 0) \ , \tag{33c}$$
$$s(0) = s_0 \ . \tag{33d}$$

Again, the optimization problem (33) is of the form (1) and will be solved by the direct multiple shooting method. Here, the focus is on the dependence of the optimal paths on the initial path ξ. In particular, this is relevant in the context of pollution control, as the pollutant has in general already been emitted before pollution control becomes affective. Due to the additional moment of inertia of delayed control problems, the past emission path has to be taken into account. In the following we show the optimal emission paths for a numerical example of the optimization problem (33) for a constant, a linear, and a cyclical initial path. We choose $V = \ln c(t)$, $\bar{l} = 1$, $\alpha = 1$, $\beta = 0.005$, $\gamma = 0.1$, $\rho = 0.03$, $t_f = 200$, $s_0 = 10$, $\xi_{const} = 1.47459$, $\xi_{lin} = 1 + 0.0815485(t + 10)$ and $\xi_{cyc} = 1.39815 + \sin[0.9\pi(t + 10)]$. To be able to compare the results for these different initial paths, they have been chosen in such a way that the stock of pollution at time $t = \sigma = 10$ is identical for all three of them ($s(10) = s_\sigma = 13$).

Figure 3 shows the optimal paths of the pollution stock and the emissions in the case of delayed stock accumulation ($\sigma = 10$) for the three different initial paths ξ. The initial paths ξ are shown as the emission paths in the time interval $t \in [-10, 0]$ in figure 3. As already mentioned earlier, the path for the pollution stock in the time interval $t \in [0, 10]$ is completely determined by the initial value s_0, the initial path ξ and the equation of motion (33b). Hence, pollution control from time $t = 0$ on only affects the pollution stock after time $t = \sigma = 10$. This shows a fundamental feature of delayed optimal control problems: the system dynamics exhibits an additional moment of inertia as the stock reacts with a delay to the control.

In all three scenarios the pollution stock rises from their initial value $s_0 = 10$ to $s_\sigma = 13$ in the time interval $t \in [0, 10]$. Nevertheless, because of the different initial paths ξ, the path of the pollution stock is concave (ξ constant), convex (ξ linear) or oscillatory (ξ cyclical). Variations in the initial path ξ cause variations in the optimal system dynamics, although the pollution stock $s_\sigma = 13$ and the long-run stationary state remains unaltered. This is best seen in the case of a cyclical initial path, which induces corresponding oscillations in the optimal emission path (figure 3 bottom).

5 Discussion

In this section we discuss the robustness of our numerical approach with respect to changes in model specifications and outline how the approach can be generalized. Furthermore, we show how our approach can be applied to numerically solve models which are discussed in the economic literature on investment gestation lags, vintage capital accumulation and habit formation.

5.1 Robustness and possible extensions of the numerical procedure

The optimization problem (1) that we discussed so far is limited in the sense that it exhibits just one state and one control variable and that the equation of motion is of a particular simple form, exhibiting just one constant delay in the control variable. In the following, we discuss how robust our approach is to more general model specifications.
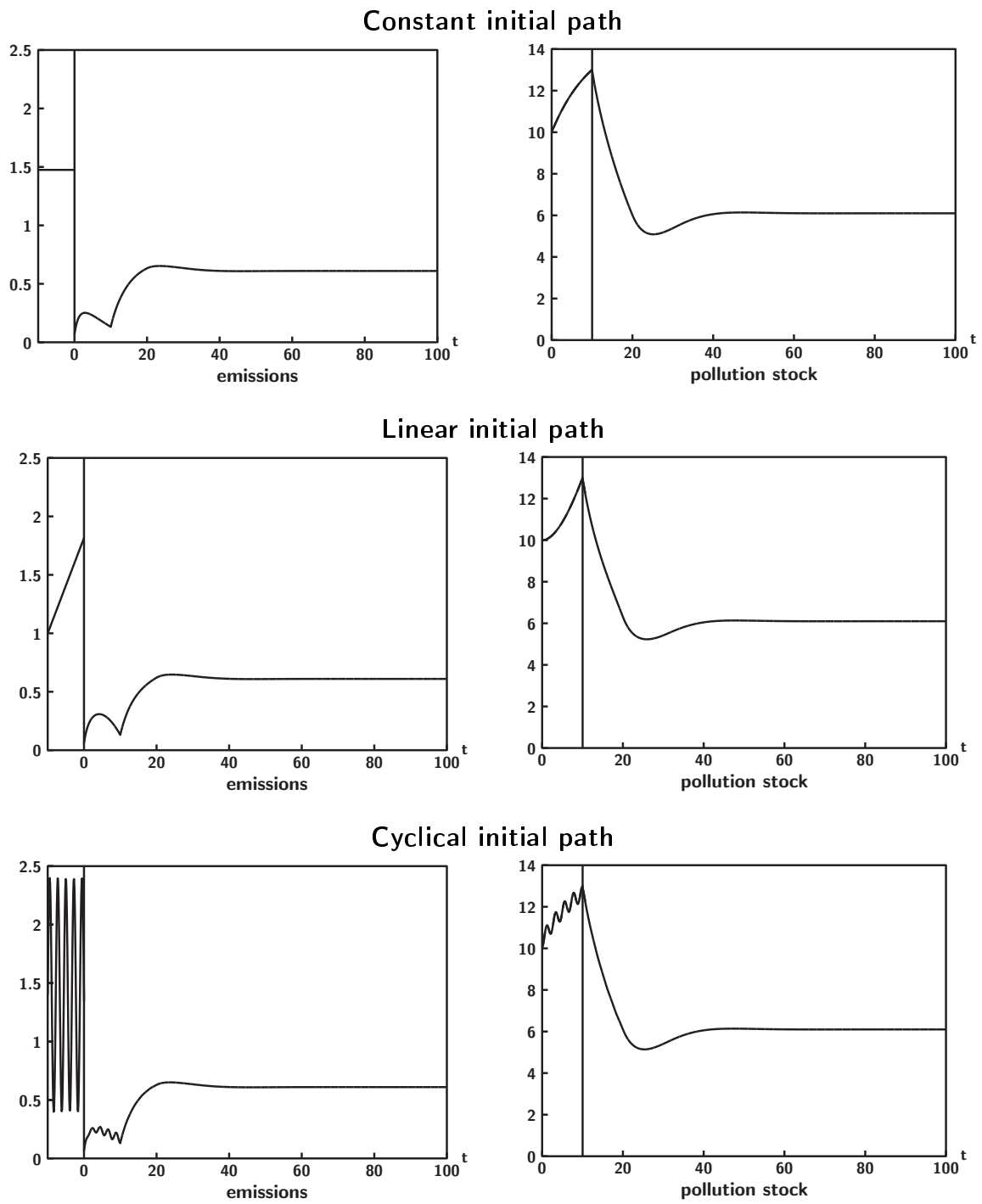
Abbildung 3: Optimal paths for the emissions (left) and the pollution stock (right) for constant (top), linear (middle) and cyclical (bottom) initial paths ξ.

Considering optimal control problems, which exhibit additional "unlagged" state and control variables poses no problem from a numerical point of view. However, computing time may increase with increasing number of state and control variables. Also the consideration of additional lagged control variables is straightforward. For each lagged control variable, we have to introduce an additional control variable and an additional constraint as described in section 3.1 problem (13), no matter if it is different control variables which exhibit one lag each or just one control variable that enters the control problem with different constant lags. However, the numerical realization requires that the different delays are multiples of one common factor.

The treatment of lagged state variables can be performed similar to the procedure described in section 3.1 problem (8), again provided that the delays in the state variables as well as in the controls are multiples of one common factor. The dimension of the resulting nonlinear optimization problems may be very large, in particular if the common factor is small compared to the time horizon t_f.

In the optimization problem (1) we assumed a particular simple equation of motion which was linear in the state and the lagged control variables. From a numerical point of view, considering non-linear equations of motion poses no additional problems as the felicity function $f(\cdot)$ is in general non-linear and, thus, we have a non-linear optimization problem anyway.

Problems with time or state-dependent delays normally cannot be reformulated in the way discussed in 3.1. For approaches to their treatment see, e.g., Bock & Schlöder (1984), where a direct approach is compared to an indirect approach resulting in nonlinear boundary value problems with retarded and advanced terms. Numerical results using a shooting method are reported.

For the aforementioned cases in which our approach is applicable, and for an increasing number of unknowns, the Newton-type based direct multiple shooting method can be expected to outperform algorithms that are built upon a componentwise optimization, as proposed, e.g., in Boucekkine *et al.* (2001) in the context of a relaxed Gauß-Seidel iteration scheme. Another advantage of our approach compared to discrete-time schemes is the possible use of fast error-controlled adaptive integrators.

5.2 Applications to economic problems with delayed problem structures

In the introduction we briefly outlined the economic literature on *investment gestation lags*, *vintage (human) capital accumulation* and *habit formation*. In the following we discuss how our approach can contribute to this literature.

Investment gestation lags

As already mentioned in the introduction, the literature on investment gestation lags can be further divided in *delivery lags* and *time-to-build*. By modeling investment as a control variable, the numeric procedure developed in this paper can directly be applied to the problem of delivery lags, as discussed by one of our examples. However, it seems that, as our approach can only handle lags in the control variables, it is not suited for

the numerical solution of time-to-build models which exhibit a delay in the state variable (e.g., Asea & Zak 1999). In fact, we cannot numerically solve Asea & Zak's (1999) model specification, but we argue that this is rather due to their specific assumption about depreciation than to the time-to-build feature in general. To see this we recall their model structure in our notation. The objective is to maximize the discounted infinite integral over instantaneous utility $u(\cdot)$ derived from consumption $c(t)$ given the following equation of motion:

$$\dot{k}(t) = f(k(t-\sigma)) - \delta k(t-\sigma) - c(t) \;, \tag{34}$$

where $k(t)$ denotes the capital stock, $f(\cdot)$ is a neoclassical production function and δ is the constant rate of depreciation. By introducing the *productive capital stock* $x(t) = k(t-\sigma)$ and investment $i(t) = f(k(t-\sigma)) - c(t)$, we can write the equation of motion (34) as:

$$\dot{x}(t) = i(t-\sigma) - \delta x(t-\sigma) \;. \tag{35}$$

In this notation, we see that Asea & Zak's (1999) model specification is in fact rather a delivery lag than a time-to-build specification with a rather unusual depreciation rule (i.e., the productive capital stock depreciates time-lagged). Applying the standard economic depreciation rule would yield an equation of motion with a delay in the control variable investment only, which is exactly of the type (1b).[6]

Moreover, we argue that our approach can be used for the analysis of more general time-to-build specifications. As mentioned in the introduction, the difference between delivery lags and time-to-build is that in the former case all investment is made in advance, while in the latter case investment is distributed over the process of creation of new capital goods. Thus, a more general time-to-build specification would be:

$$\dot{x}(t) \;=\; i(t-\sigma) - \delta x(t) \;, \tag{36a}$$

$$c(t) \;=\; f(x(t)) - \int_{t-\sigma}^{t} m(t-s)i(s)\,ds \;. \tag{36b}$$

The interpretation is straightforward. The creation of capital goods needs the fixed time-span σ. Denoting by $i(t)$ the amount of new capital goods of which the production started at time t and assuming depreciation of the capital stock at the constant rate δ, the accumulation of capital is governed by the delayed differential-difference equation (36a). The function $m(t)$, with carrier $[0, \sigma]$, denotes the resource input needed at time t for new capital goods which were started to produce at time 0. Assuming only one commodity that can be both consumed and used for capital production, we achieve equation (36b). By discretization of the integral in equation (36b), we can achieve a form which is solvable by our numerical approach. If we divide the production process of new capital, which needs the time-span σ, into N steps, each of the same duration $\frac{\sigma}{N}$,

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[6] In their introduction Asea & Zak (1999) justify their model specification by analytical tractability. Ironically, their specification (34) is easier to analyze *analytically*, while it poses more difficulties *numerically*.

we can write (36b):

$$c(t) \approx f(x(t)) - \sum_{n=0}^{N-1} m_n i \left(t - \sigma + n\frac{\sigma}{N} \right) \frac{\sigma}{N} \,, \tag{37}$$

where $m_n = m \left(\sigma - n\frac{\sigma}{N} \right)$ is the amount of resource input needed at the time $n\frac{\sigma}{N}$ of the production process of new capital goods. Thus, we achieve an optimal control problem with one stock and one control variable, where the control variable appears with N different but constant lags.

Vintage (human) capital accumulation

For the sake of simplicity we only consider physical capital. However, there is a strong formal correspondence between vintage physical and vintage human capital (compare, e.g., Boucekkine *et al.* 2004).

As outlined by Benhabib & Rustichini (1991), vintage capital models can be characterized by general, non-exponential rates of depreciation, which can include learning by using or gestation lags. Denoting the productive capital stock at time t by $k(t)$ and investment at time t by $i(t)$, the objective is once again to maximize the discounted infinite integral over instantaneous utility $u(\cdot)$ derived from consumption $c(t)$, where $c(t) = f(k(t)) - i(t)$, with $f(\cdot)$ being a neoclassical production function. The capital stock $k(t)$ is given by:

$$k(t) = \int_{-\infty}^{t} i(s)m(t-s)\, ds \,, \tag{38}$$

where $m(t)$ $(t \geq 0)$ denotes the depreciation schedule. Differentiating with respect to time yields the following equation of motion:

$$\dot{k} = \int_{-\infty}^{t} i(s)\frac{d}{dt}\left(m(t-s)\right)\, ds + i(t)m(0) \,. \tag{39}$$

The specification with a constant rate of depreciation δ is achieved by setting $m(t) = \exp[-\delta t]$.

In the case that capital does not depreciate but has a constant lifetime σ (i.e., the one-hoss shay depreciation), $m(t) = \theta(\sigma - t)$, with $\theta(t)$ the Heaviside step function (i.e., $\theta(t) = 1$, if $t \geq 0$, and $\theta(t) = 0$, else) and, thus, the equation of motion yields

$$\dot{k}(t) = i(t) - i(t - \sigma) \,, \tag{40}$$

which results in an optimal control problem that can be solved directly by our numerical algorithm.

In the general case of equation (39), we can approximate the integral analogously to the case of investment gestation lags, if $\lim_{t \to -\infty} \frac{d}{dt}m(t) = 0$:

$$\begin{aligned}
\dot{k}(t) &= i(t)m(0) + \int_{-\infty}^{t-\sigma} i(s)\frac{d}{dt}\left(m(t-s)\right)\, ds + \int_{t-\sigma}^{t} i(s)\frac{d}{dt}\left(m(t-s)\right)\, ds \\
&\approx i(t)m(0) + \int_{t-\sigma}^{t} i(s)\frac{d}{dt}\left(m(t-s)\right)\, ds \,, \quad \text{for } \sigma \text{ sufficiently large} \,.
\end{aligned} \tag{41}$$

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The integral in equation (41) is of the same form as the integral in equation (36b) and can be discretized analogously yielding

$$\dot{k}(t) \approx i(t)m(0) + \sum_{n=0}^{N-1} \dot{m}_n i\left(t - \sigma + n\frac{\sigma}{N}\right)\frac{\sigma}{N} \ , \tag{42}$$

where $\dot{m}_n = \frac{d}{dt}m(t)\big|_{(\sigma - n\frac{\sigma}{N})}$. Again, we achieve an optimal control problem with one stock and one control variable, where the control variable appears with N different but constant lags.

Habit formation

In models of habit formation, instantaneous utility $u(\cdot)$ is derived not only from consumption at time t but also depends on some stock of habits $h(t)$. In general, instantaneous utility depends negatively on the stock of habits (i.e., $\frac{\partial u}{\partial h} < 0$). As an example consider the specification of instantaneous utility of Carroll *et al.* (2000):

$$u(c(t), h(t)) = \frac{\left(\frac{c(t)}{h(t)^\gamma}\right)^{1-\theta}}{1 - \theta} \ , \tag{43}$$

where θ is the coefficient of relative risk aversion and γ measure how much weight is given to the absolute level of consumption in comparison to the consumption level relative to the habit stock.

The habit stock is some general mean of past consumption levels. In the most general form we can write $h(t)$ as

$$h(t) = \int_{-\infty}^t c(s)m(t - s)\,ds \ , \tag{44}$$

where $m(t)$ denotes the weighting function. Obviously, equation (44) is formally identical to equation (38) and, thus, following the same line of argument all weighting functions $m(t)$ with $\lim_{t \to -\infty} \frac{d}{dt}m(t) = 0$ can be approximated in a way to be numerically solvable with our approach.

A special case, for which our numerical algorithm is directly applicable, is achieved by the weighting function $m(t) = \frac{1}{\sigma}\theta(\sigma - t)$. This is the direct analogon to the one-hoss shay depreciation rule in the vintage capital context and means that the habit stock at time t is the average of consumption over the interval $[t - \sigma, t]$, which yields the following equation of motion for the habit stock:

$$\dot{h} = \frac{1}{\sigma}\left[c(t) - c(t - \sigma)\right] \ . \tag{45}$$

6 Conclusions

As well known from the literature, delayed optimal control problems with one stock and one control variable exhibit in general a qualitatively different system dynamics

compared to instantaneous optimal control problems. While the optimal paths of the latter converge strictly monotonically towards the stationary state, the former exhibit oscillatory and exponentially damped optimal paths.

In this paper, we have drawn attention to the numerical solution of optimal control problems with a delay in the control variable. We have shown how a simple delayed optimal control problem can be reformulated such that direct state-of-the-art methods can be applied. In particular, we presented two different problem reformulations and compared the performance of Bock's direct multiple shooting algorithm, implemented in the software package `MUSCOD-II`. While the first reformulation increases the dimensionality of the resulting optimization problem drastically by introducing as many new stock and control variables as the time horizon t_f, over which is optimized, is a multiple of the time-lag σ, the second reformulation only introduces one additional control variable, irrespective of the time horizon t_f and the time-lag σ. While the latter reformulation exhibits better computational performance, the former allows for intuitive explanations of some standard analytic results of the control-delayed optimal control problem.

Numerical optimization plays a crucial part in the analysis and understanding of delayed optimal control problems, as even the linear approximation of the system dynamics around the stationary state is not analytically tractable. As we understand the lack of application of delayed optimal control in economics to be (at least partly) a consequence of the analytical and numerical difficulties, we hope that this paper encourages broader research in this area. In fact, there a numerous applications in the field of economics alone. With two examples we have shown how to apply the method for the rigorous analysis of the transition from instantaneous to delayed capital accumulation and for the analysis of the influence of the initial path on the optimal time-lagged accumulation of a pollution stock. Further, we have discussed how general investment gestation lag, vintage capital accumulation and habit formation models can be reformulated to be tractable by our algorithm. However, we also expect our numeric approach to be valuable for other fields of scientific endeavor.

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